Counting smaller trees in the Tamari order

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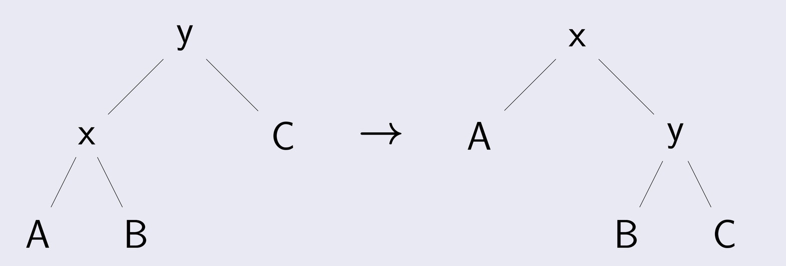
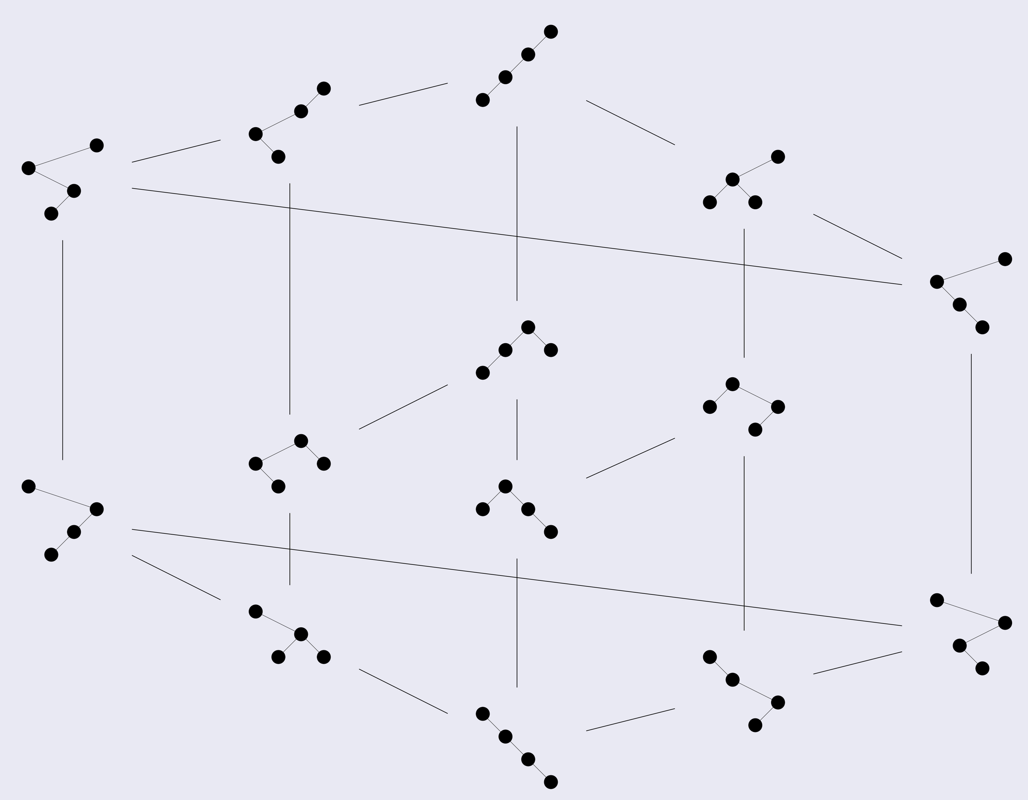
The right rotation on binary trees



The right rotation is the cover relation of the Tamari order on binary trees.

The Tamari lattice



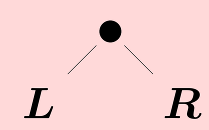
Main result

Given a binary tree T, we define its *Tamari polynomial* $\mathcal{B}_T(x)$ by:

$$\mathcal{B}_\emptyset := 1$$

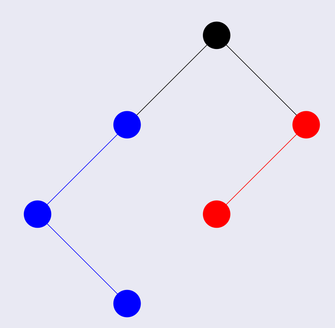
$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

with T =



$\mathcal{B}_T(x)$ counts the number of trees smaller than or equal to T in the Tamari order according to the number of nodes on their left border. In particular, $\mathcal{B}_T(1)$ is the number of trees smaller than T.

Example

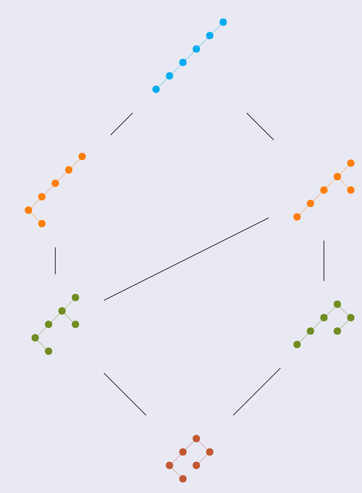


$$\mathcal{B}_T(x) = x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x)-\mathcal{B}_R(1)}{x-1}$$

$$\mathcal{B}_T(x) = x(x^2 + x^3)(1 + x + x^2)$$

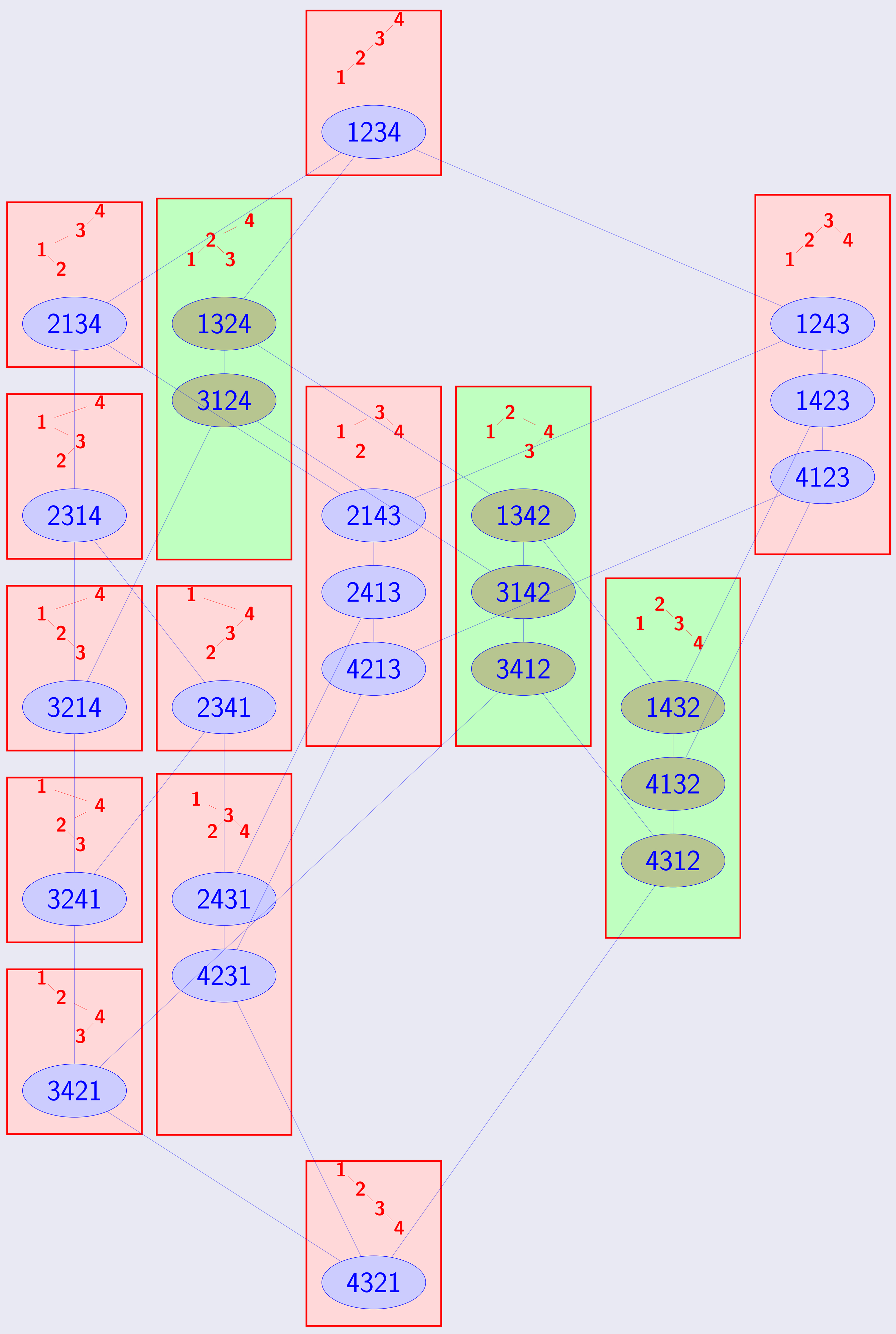
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$

$$\mathcal{B}_T(1) = 6$$

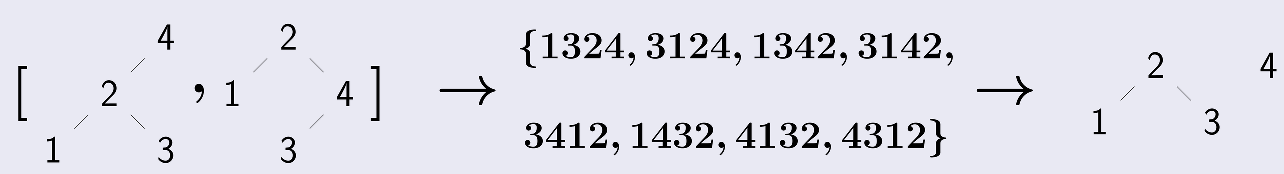


The Tamari lattice is a quotient of the weak order

Each binary tree has a unique labelling as *binary search tree*. The linear extensions of these labelled binary trees are intervals of the weak order: the *sylvester classes*.

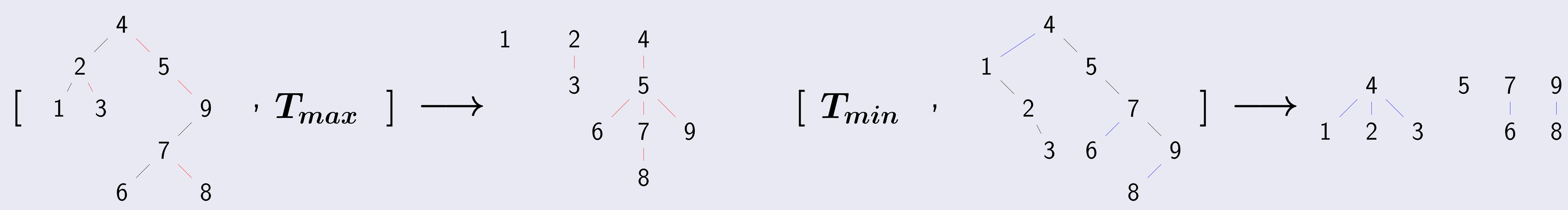


An interval of the Tamari lattice can be seen as a union of sylvester classes. It is represented as a poset whose linear extensions correspond to these sylvester classes.

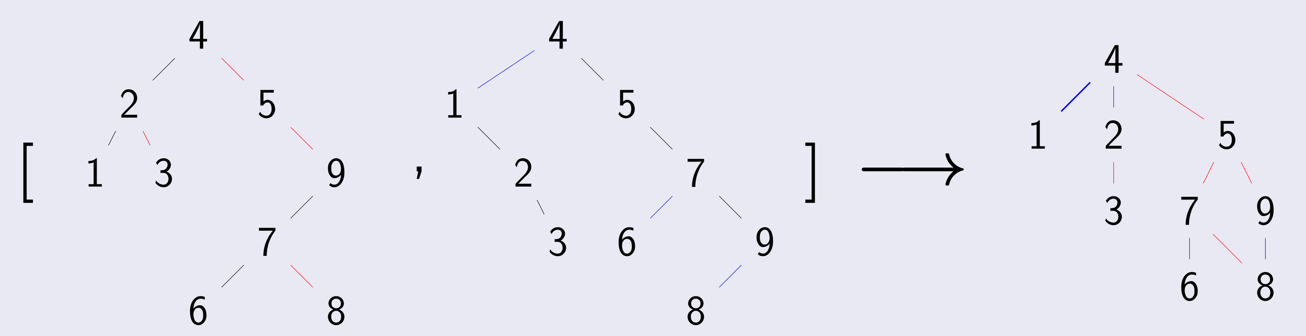


Tamari Interval-posets

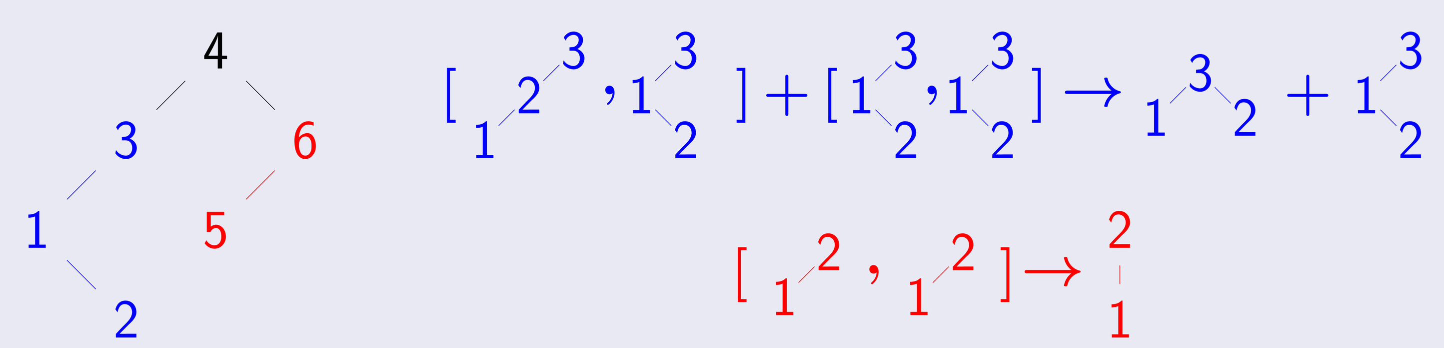
An *interval-poset* is a poset representing an interval of the Tamari lattice. With each binary tree T, one can associate an increasing and a decreasing forest. They correspond respectively to the initial interval $[T_{min}, T]$ and the final interval $[T, T_{max}]$ of the Tamari lattice.

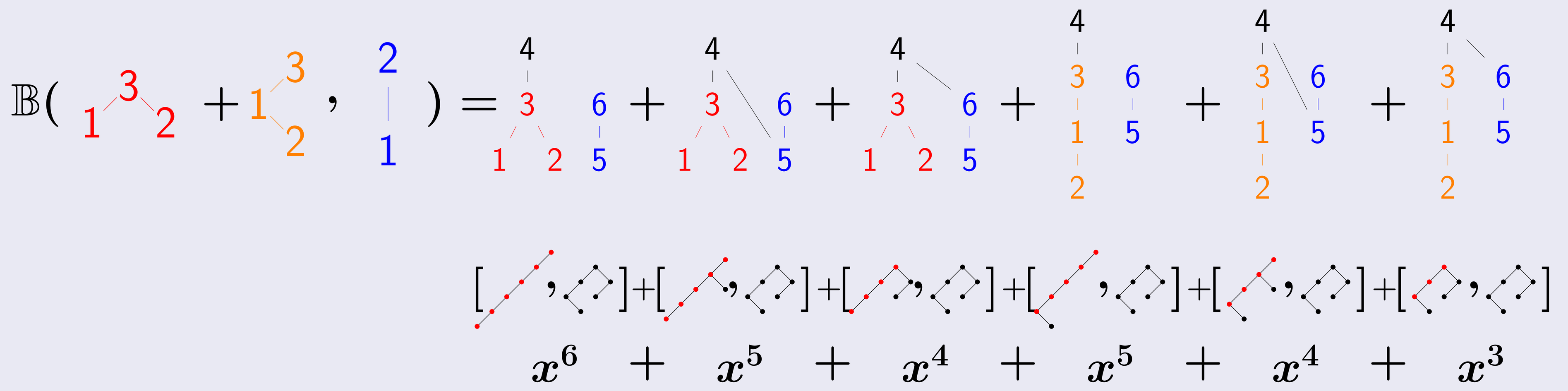


To construct the interval-poset $[T_1, T_2]$, we combine the decreasing forest of T_1 and the increasing forest of T_2.



The number of trees smaller than or equal to T is the number of intervals $[T', T]$ having T as maximal element. Our proof is based on a combinatorial interpretation of the bilinear operation $\mathcal{B}_T$.





Other results and perspectives

New proof of the number of intervals of the Tamari lattice, q-generalization of the main result, generalization to m-Tamari, relations with flows of rooted trees, …