Treillis de Tamari et ordre faible

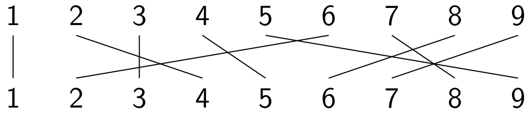
Viviane Pons

Universität Wien

Nice, 20 mars 2014

Groupe des permutations

$\sigma = 143592867$



Groupe des permutations

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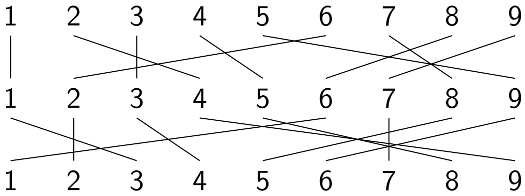

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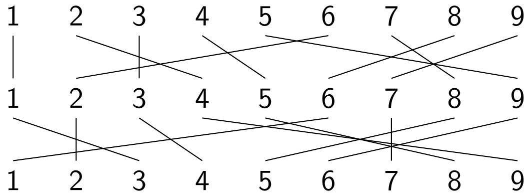

$$\mu.\sigma = 394862517$$

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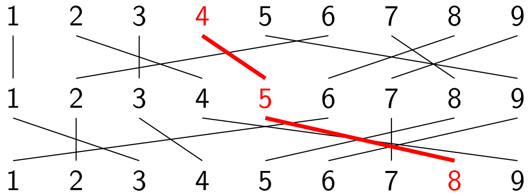

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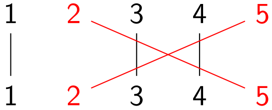
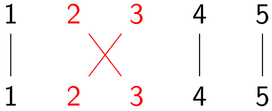
Transpositions



$(2, 5)$

Transpositions simples



$(2, 3) = s_2$

Ordre faible droit

σ

σs_i

$$\ell(\sigma s_i) = \ell(\sigma) + 1$$

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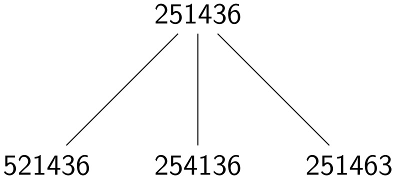

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251436

521436 254136 251463

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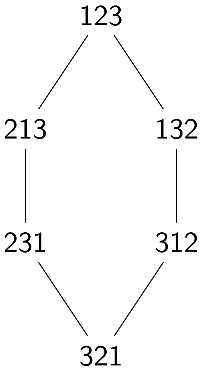
Ordre faible droit

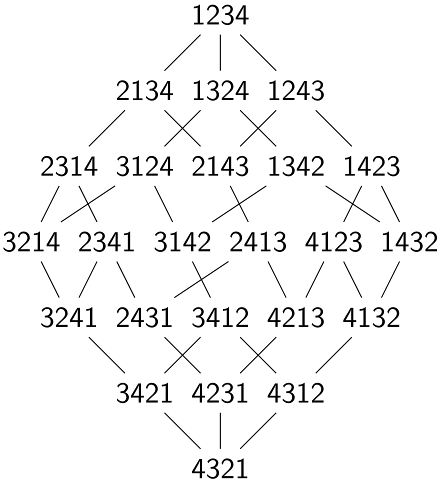
$$\begin{tikzcd} \sigma \arrow[d, dash] \\ \sigma s_i \end{tikzcd}$$

$$\ell(\sigma s_i) = \ell(\sigma) + 1$$

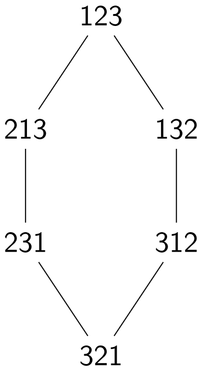
$$\begin{tikzcd} & 251436 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ 521436 & 254136 & 251463 \end{tikzcd}$$

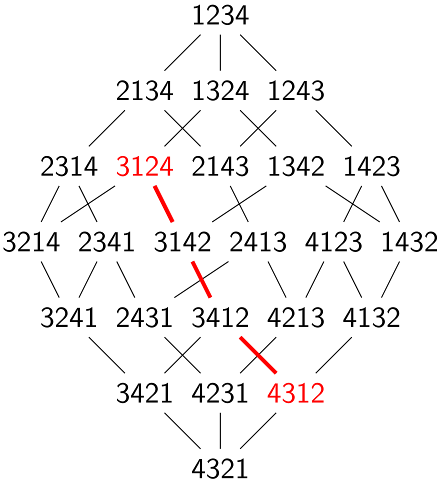
Ordre faible droit



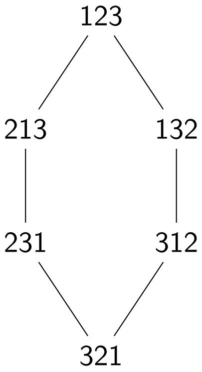


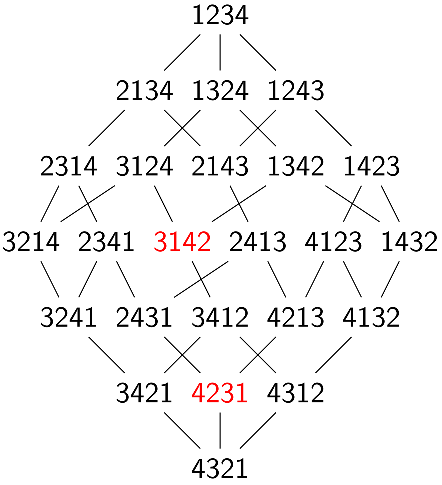
Ordre faible droit



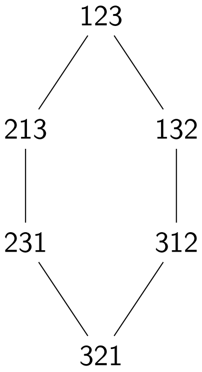


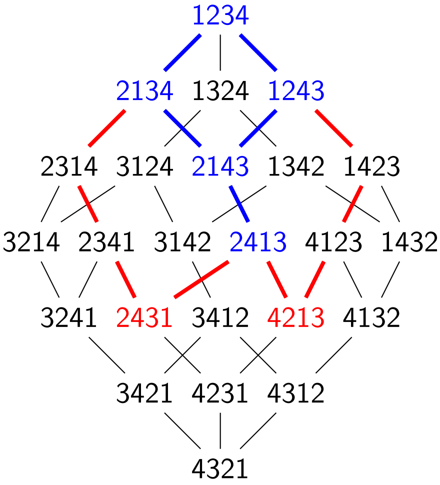
Ordre faible droit



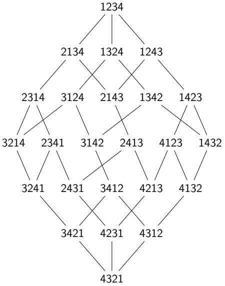


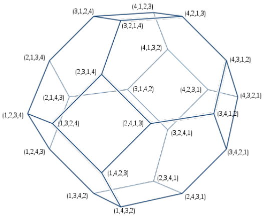
Ordre faible droit

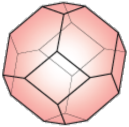




Permutoèdre







Algèbre de Malvenuto Reutenauer

Éléments de base indexés par les permutations : $(\mathbf{F}_\sigma)$.
Produit définit par le *shuffle* :

$$\mathbf{F}_{21} . \mathbf{F}_{12} = \mathbf{F}_{21 \,\overline{\sqcup\!\sqcup}\, 12}$$
$$=$$

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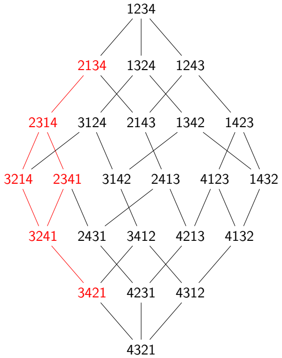
$$



$$\mathbf{F}_{21} . \mathbf{F}_{12} = \mathbf{F}_{21 \shuffle 12}$$
$$= \mathbf{F}_{2134} + \mathbf{F}_{2314} + \mathbf{F}_{2341} + \mathbf{F}_{3214} + \mathbf{F}_{3241} + \mathbf{F}_{3421}$$

Treillis de Tamari

- 1962, Tamari : ordre sur les parenthésages formels
- 1972, Huang, Tamari : structure de treillis

Treillis de Tamari

- 1962, Tamari : ordre sur les parenthésages formels
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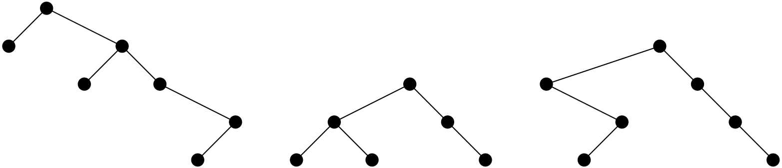
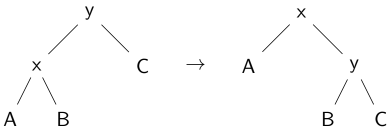
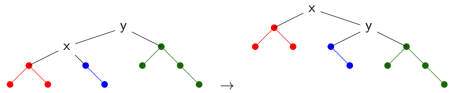
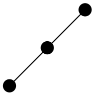
Arbres binaires



Rotation droite







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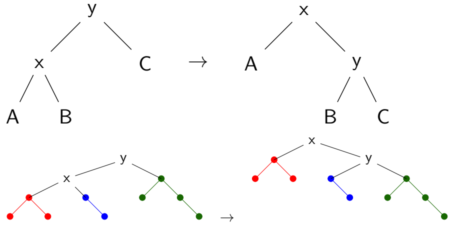

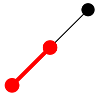

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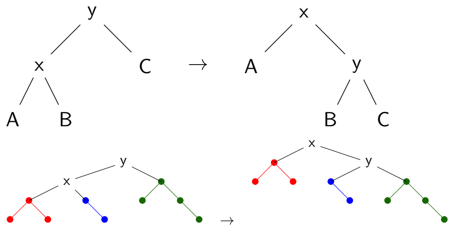

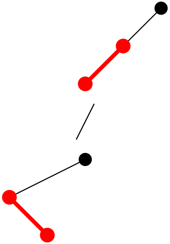

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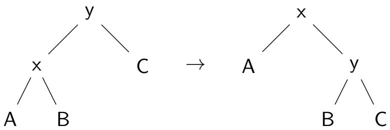

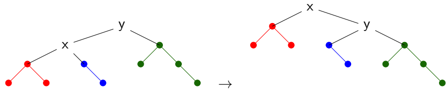

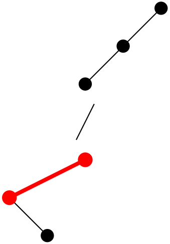

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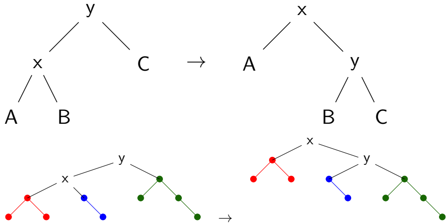

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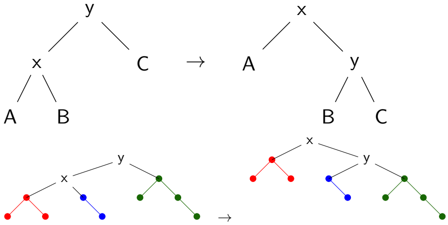

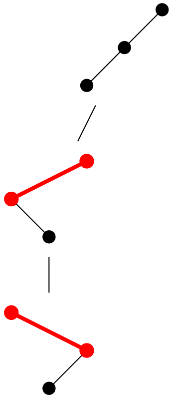

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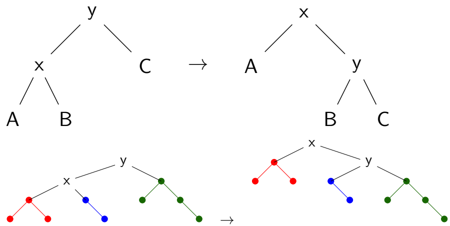

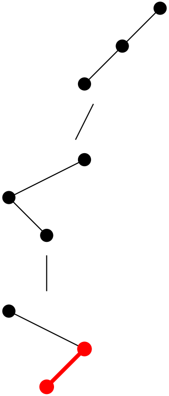

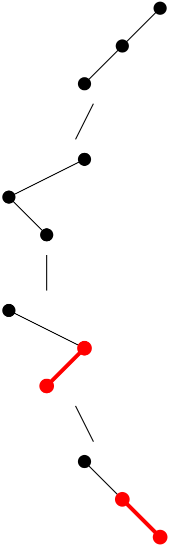

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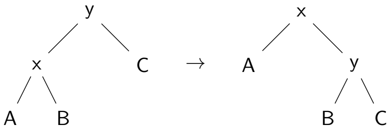
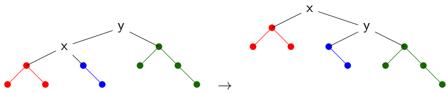
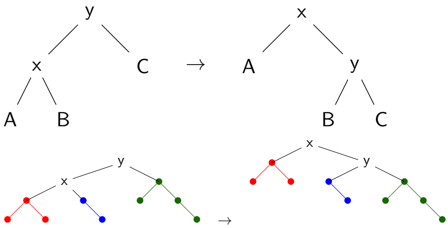
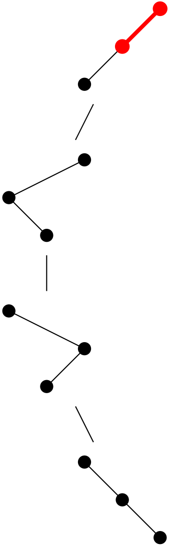




Rotation droite





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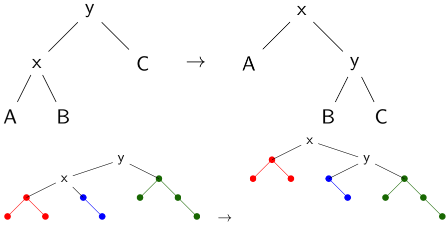

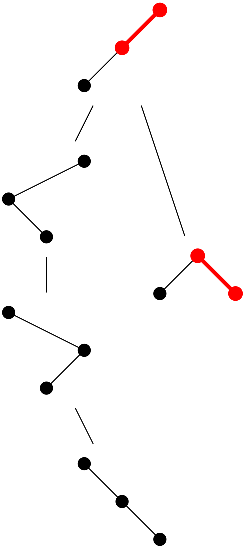

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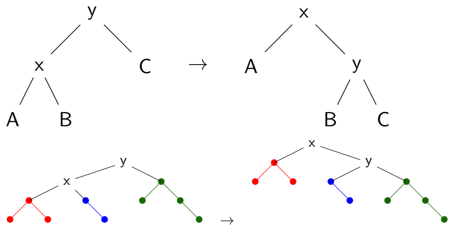
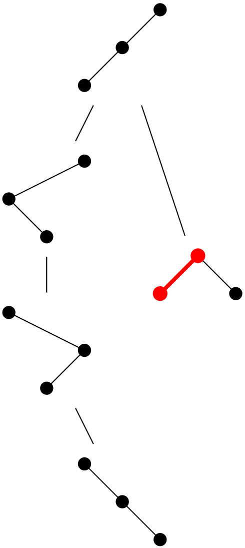
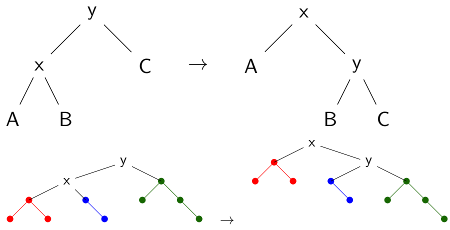
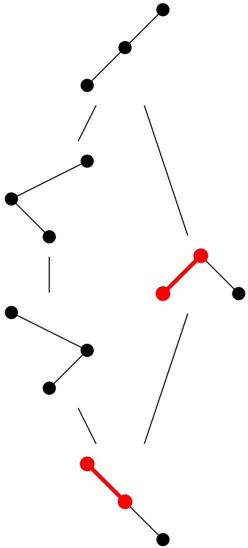




Rotation droite





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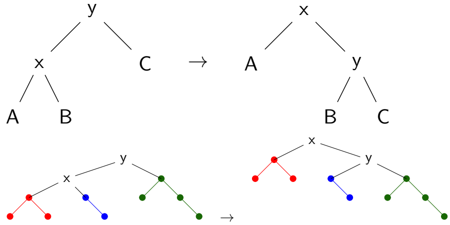
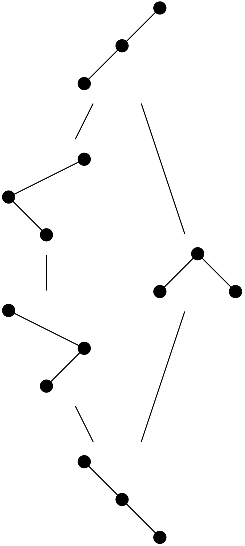
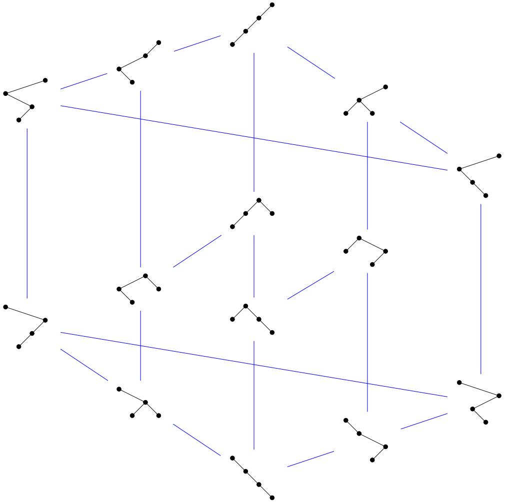
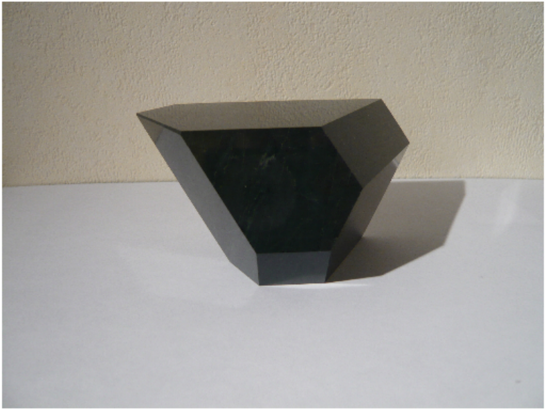


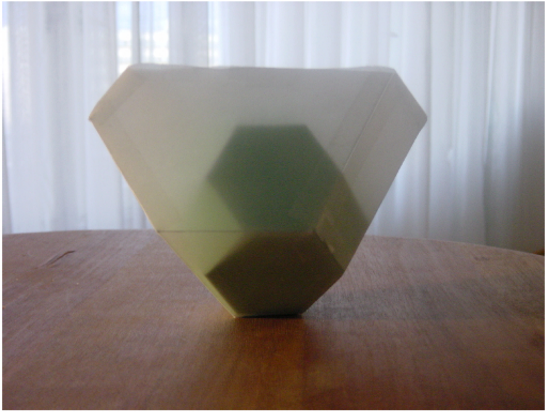




Associaèdre

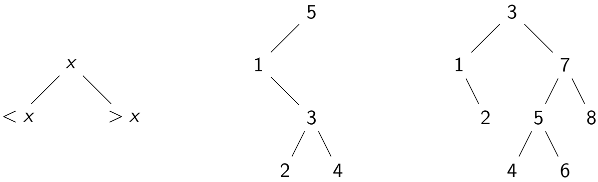


Associaèdre et permutoèdre



Lien avec l'ordre faible

Étiquetage canonique

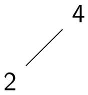


Insertion dans un arbre binaire de recherche

$$1532\textcolor{red}{4} \quad \longrightarrow \quad 4$$

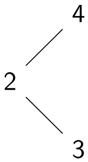
Insertion dans un arbre binaire de recherche

$15324 \rightarrow$



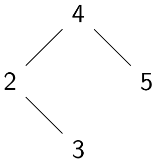
Insertion dans un arbre binaire de recherche

15324 $\rightarrow$



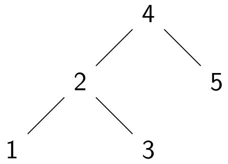
Insertion dans un arbre binaire de recherche

$1\textcolor{red}{5324} \quad \rightarrow$



Insertion dans un arbre binaire de recherche

15324 $\rightarrow$



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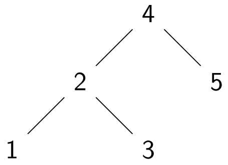
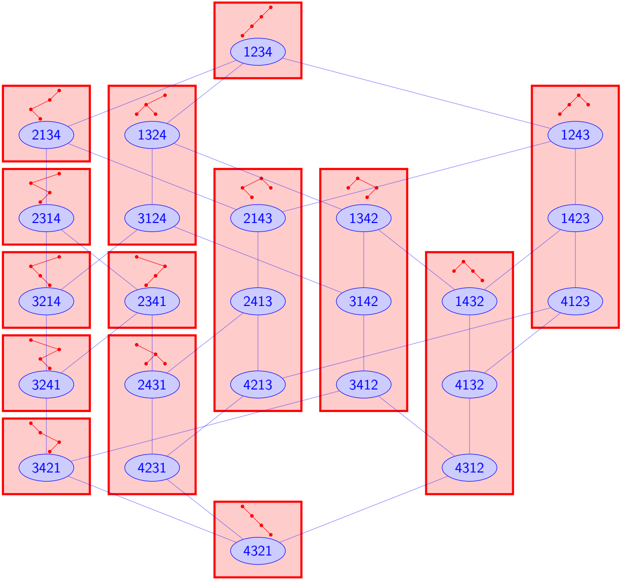


Caractérisation : les permutations qui correspondent à un arbre donné sont ses extensions linéaires
$15324, 31254, 35124, 51324, \ldots$

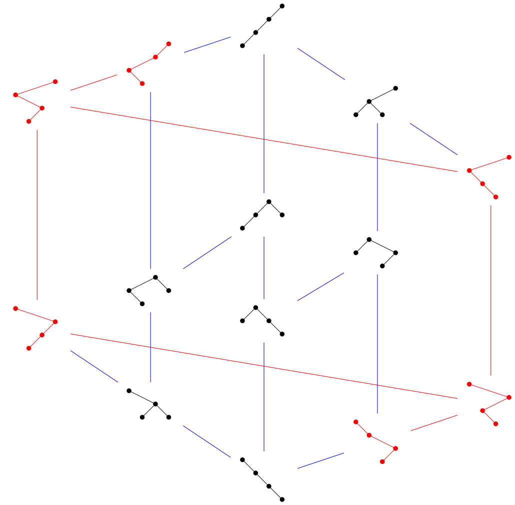

1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321

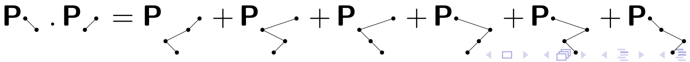
Algèbre sur les arbres binaires

$$\mathbf{P}_T = \sum_{\mathrm{ABR}(\sigma)=T} \mathbf{F}_\sigma$$

$\mathbf{P}$  $= \mathbf{F}_{2143} + \mathbf{F}_{2413} + \mathbf{F}_{4213}$

- Loday, Ronco, 1998.
- Hivert, Novelli, Thibon, 2005.





Intervalles du treillis de Tamari

- Énumération : Chapoton 2007

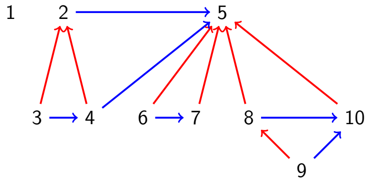
$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$$

- Bijection avec les triangulations : Bernardi, Bonichon 2009
- Bijection avec des flots sur les forêts : Chapoton, Châtel, P., 2013

Intervalles-posets

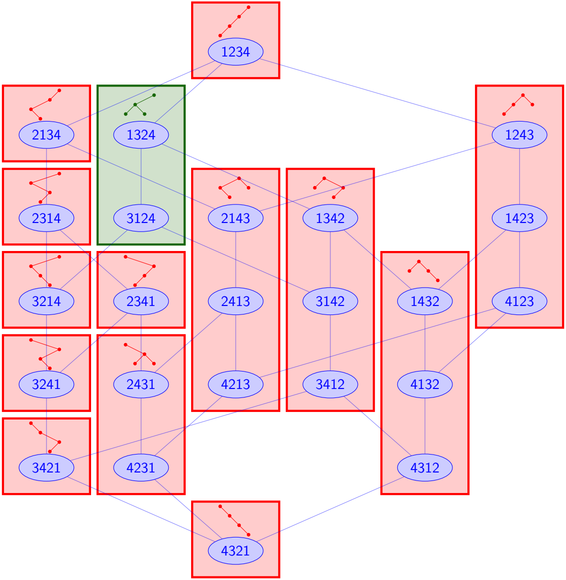
Un intervalle-poset est un poset de taille n étiqueté par $1, \ldots, n$ tel que :

- si $a < c$ et a précède c alors b précède c pour tout $a < b < c$;
- si $a < c$ et c précède a alors b précède a pour tout $a < b < c$.



Theorem (2013 – Châtel, P.)

Les intervalles-posets sont en bijection avec les intervalles du treillis de Tamari.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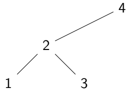
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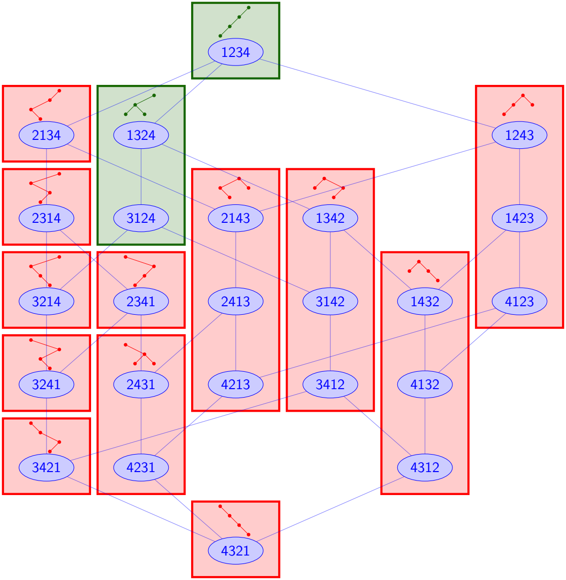
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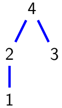
$F_{\leq}(T)$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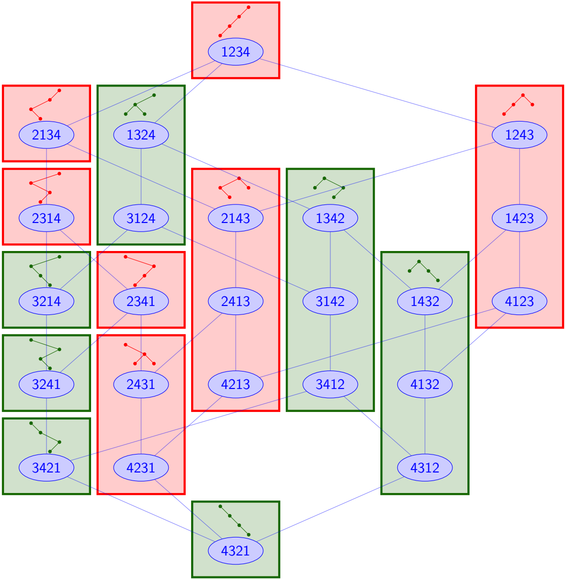
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$F_{\geq}(T)$


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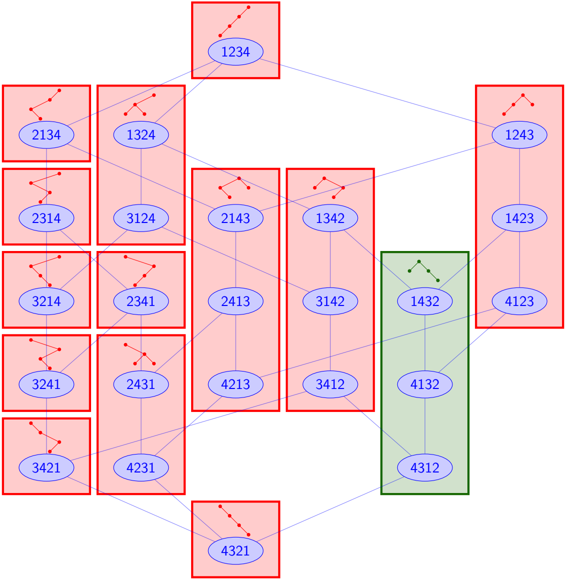
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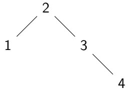
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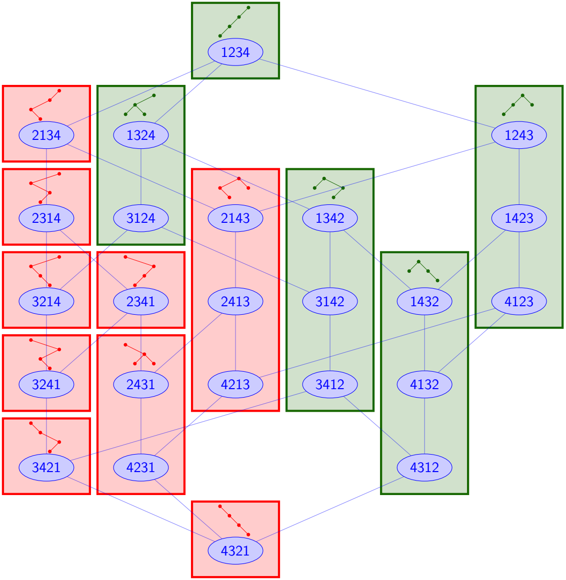
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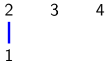
$$F_{\leq}(T')$$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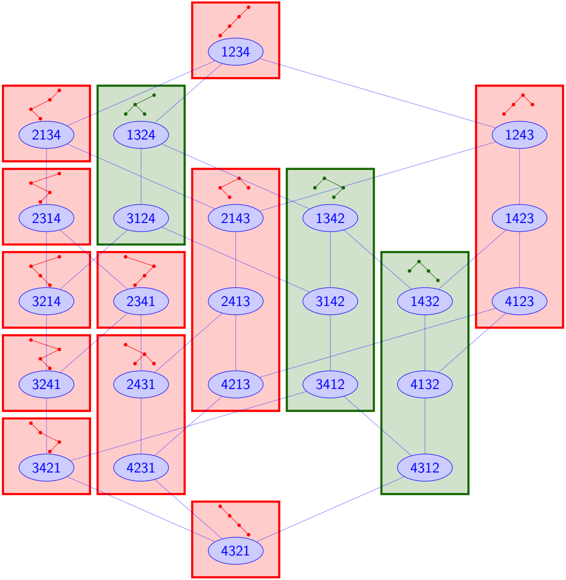
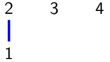


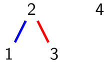
$F_{\geq}(T)$

$F_{\leq}(T')$



Intervalle-poset

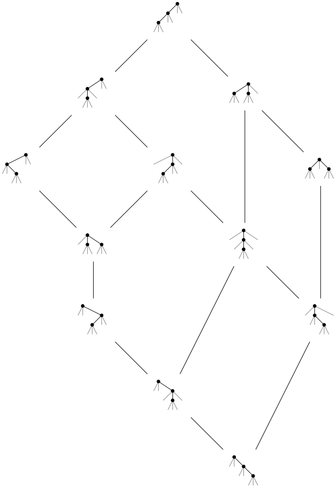
$[T, T']$



Treillis de m-Tamari

- Bergeron, Préville-Ratelle : posets de m-Tamari
- Bousquet-Mélou, Fusy, Préville-Ratelle : nombre d'intervalles

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n+m}{n-1}$$



Treillis Cambriens

- introduits par Reading, 2006
- un treillis sur des arbres binaires avec "signature"
- Permutations signées, algèbres cambriennes