

Intervals of the Tamari lattice

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UQAM, May 2nd, 2014

Tamari lattice

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- ▶ 1972, Huang, Tamari : lattice structure
- ▶ 2007, Chapoton : number of intervals

$$\frac{2}{n(n+1)} \binom{4n+1}{n-1}$$

m -Tamari lattices

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- Bergeron, Préville-Ratelle : m -Tamari posets

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- ▶ Bergeron, Préville-Ratelle : m -Tamari posets
- ▶ Bousquet-Mélou, Fusy, Préville-Ratelle : lattice structure and number of intervals

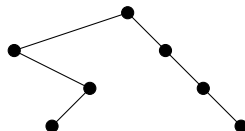
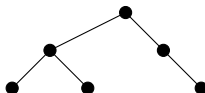
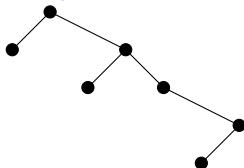
$$\frac{m+1}{n(mn+1)} \binom{(m+1)^2 n + m}{n-1}$$

Binary trees

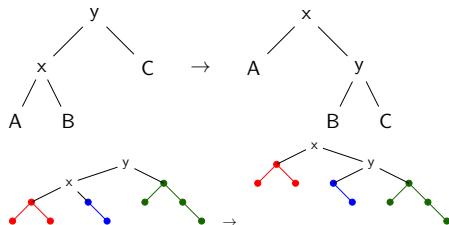
Recursive definition :

- ▶ the empty tree or
- ▶ a left subtree and a right subtree grafted to a root node

Examples

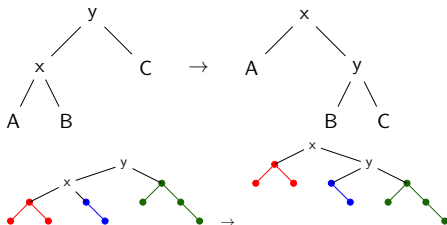


Right rotation



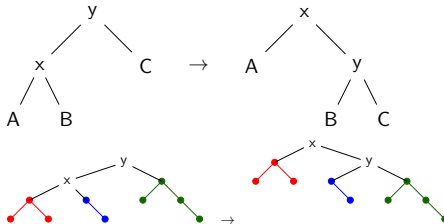


Right rotation

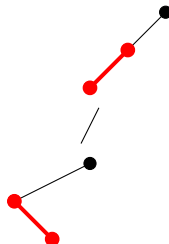
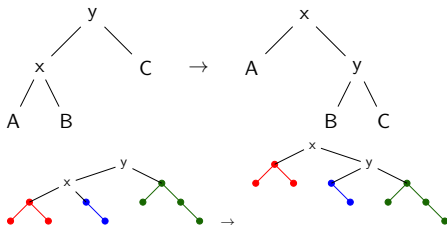




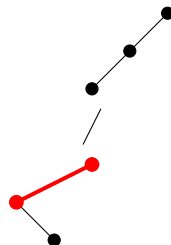
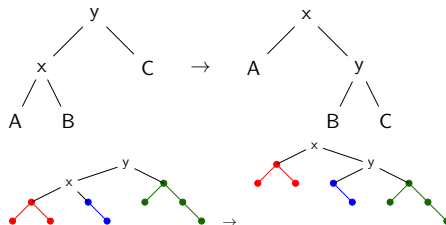
Right rotation



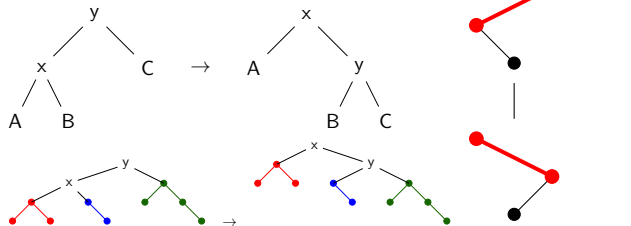
Right rotation



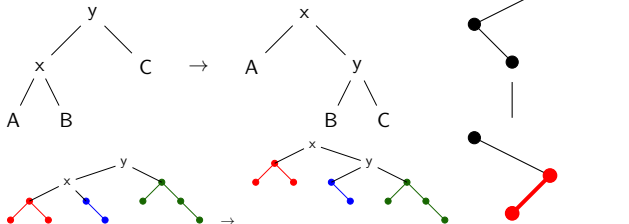
Right rotation



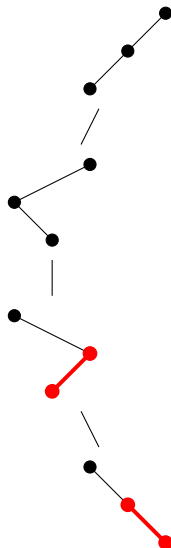
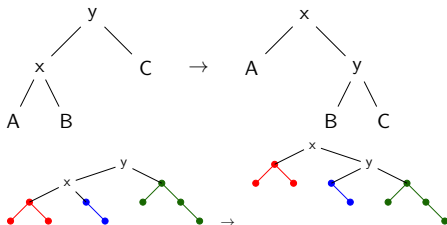
Right rotation



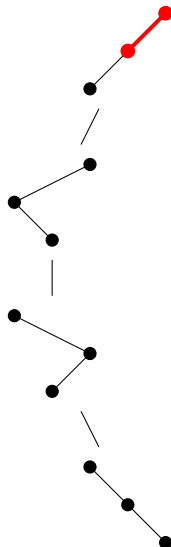
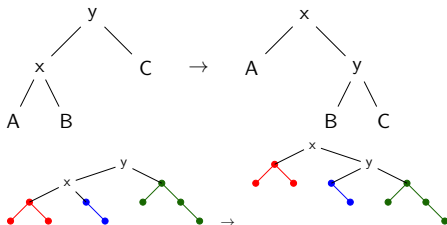
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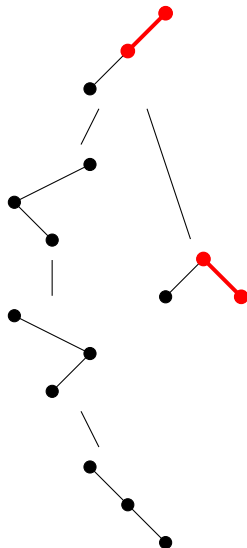
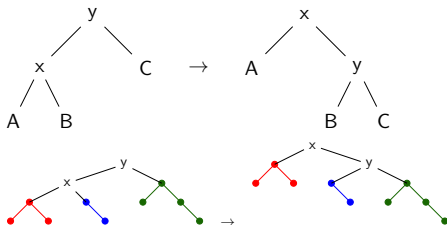
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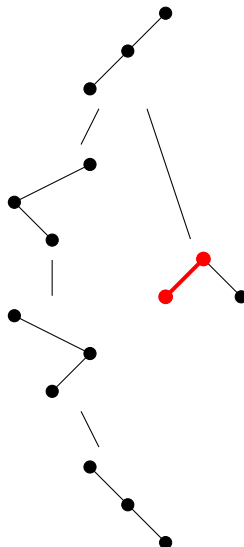
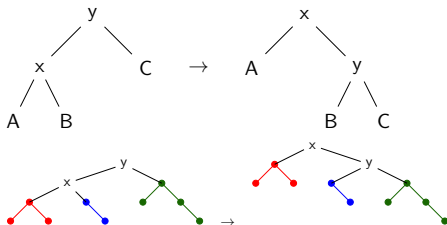
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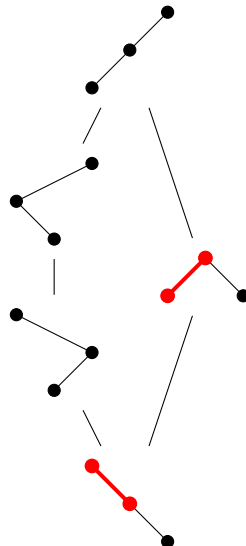
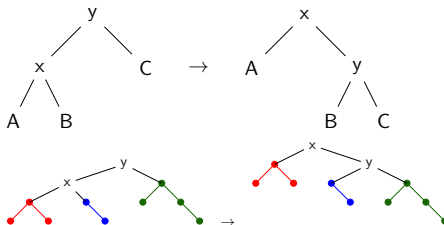
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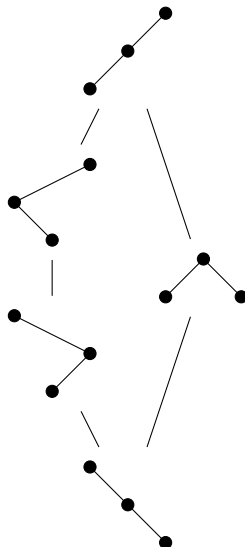
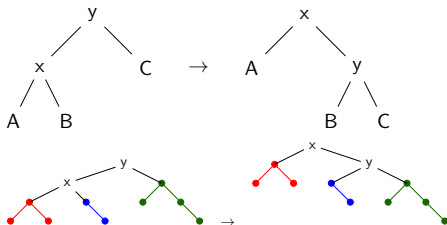
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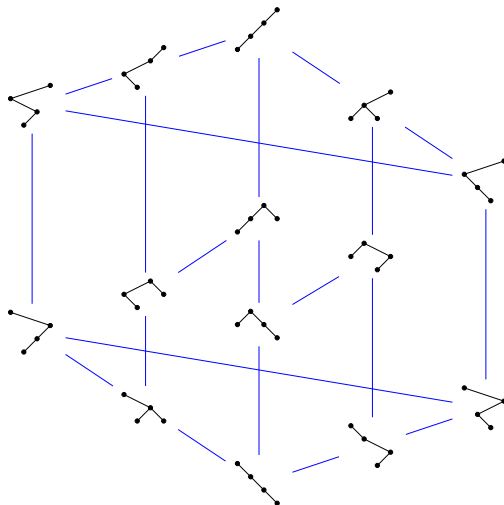


Right rotation

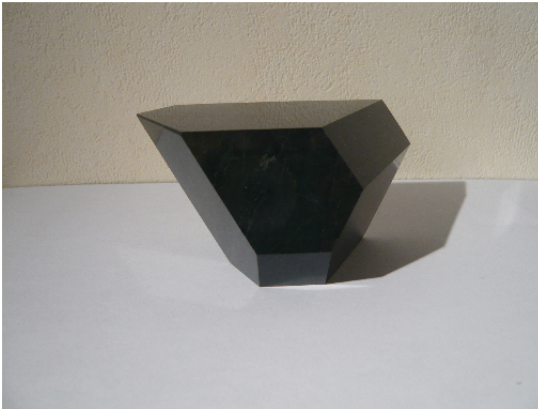


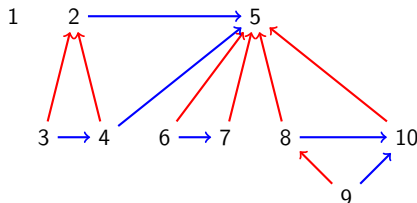
Right rotation





Associahedron



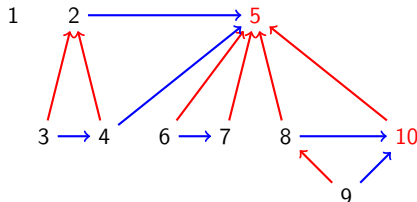


Definition

An interval-poset is a poset of size n , labelled with $1, \dots, n$ such that

- ▶ if $a < c$ and $c \triangleleft a$ then $b \triangleleft a$ for all $a < b < c$,
- ▶ if $a < c$ and $a \triangleleft c$ then $b \triangleleft c$ for all $a < b < c$.

We write $a \triangleleft b$ for a lower than b in the poset.

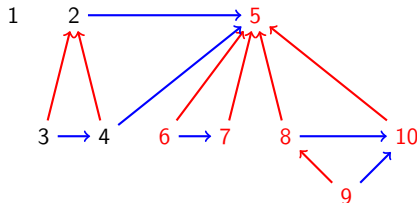


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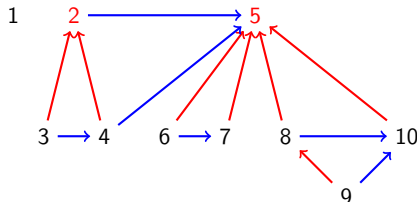


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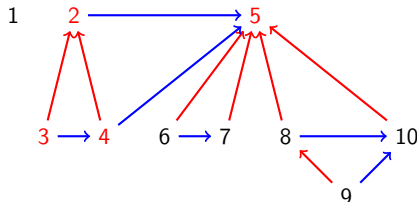


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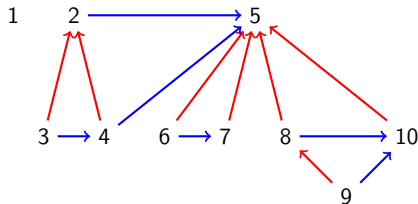


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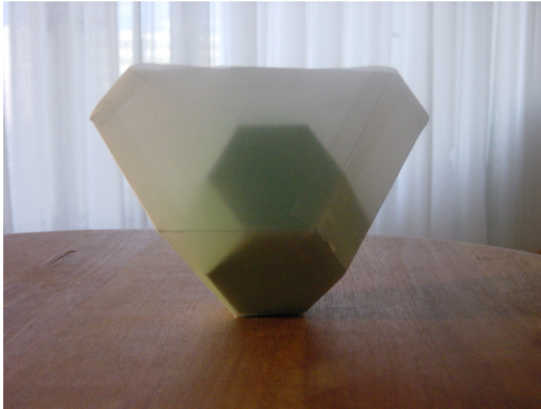
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Theorem (Châtel, P.)

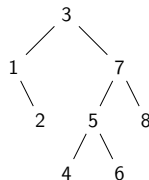
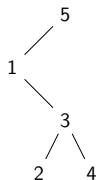
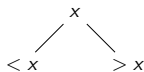
Interval-posets are in bijections with intervals of the Tamari lattice.

Link with the weak order



(image from Jean-Louis Loday)

Binary search tree

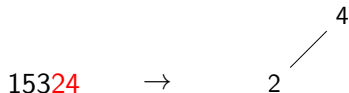


Binary search tree insertion

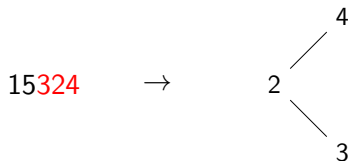
15324 \rightarrow

4

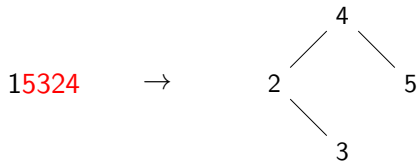
Binary search tree insertion



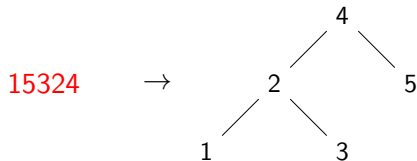
Binary search tree insertion



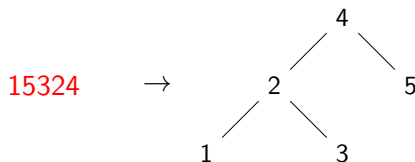
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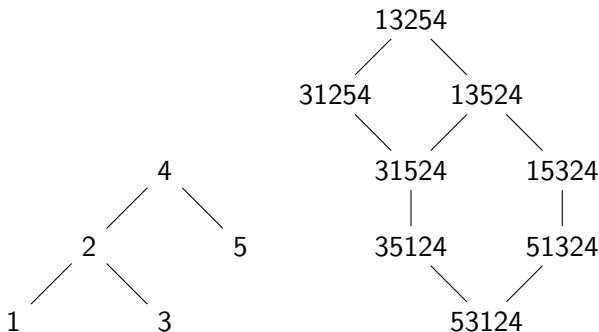
Binary search tree insertion

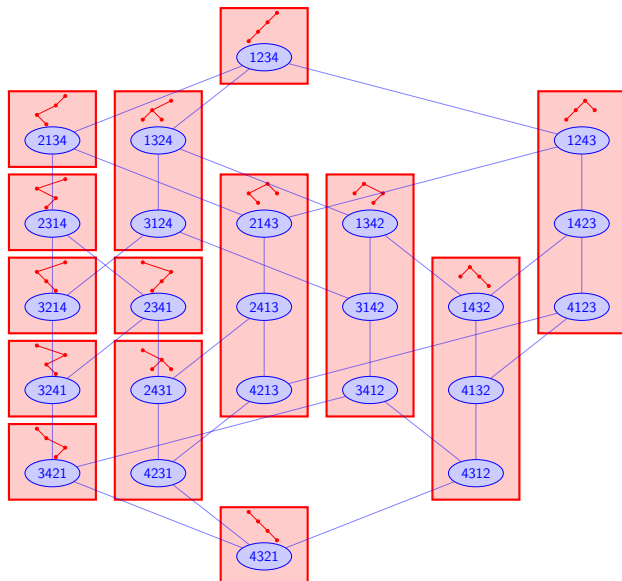


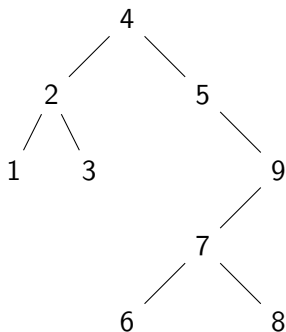
Characterization : the permutations sent to a given tree are its linear extensions

15324, 31254, 35124, 51324, ...

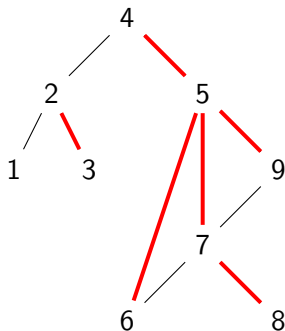
Binary search tree insertion

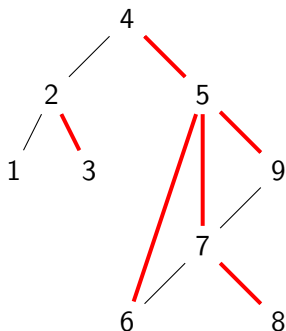




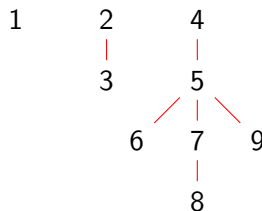


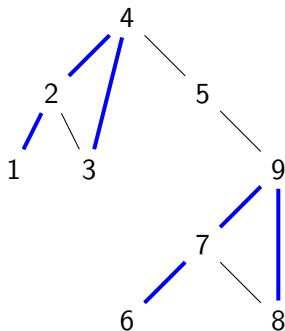
final forest $F_{\geq}(T)$



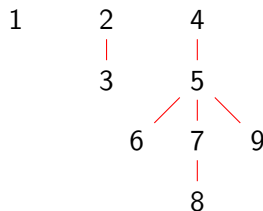


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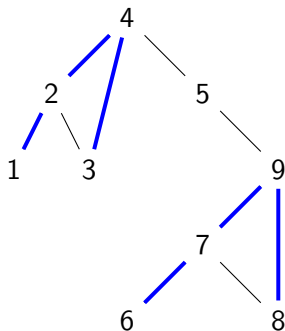




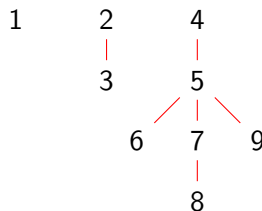
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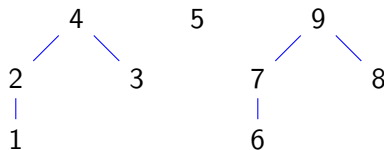
Initial forest $F_{\leq}(T)$

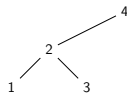
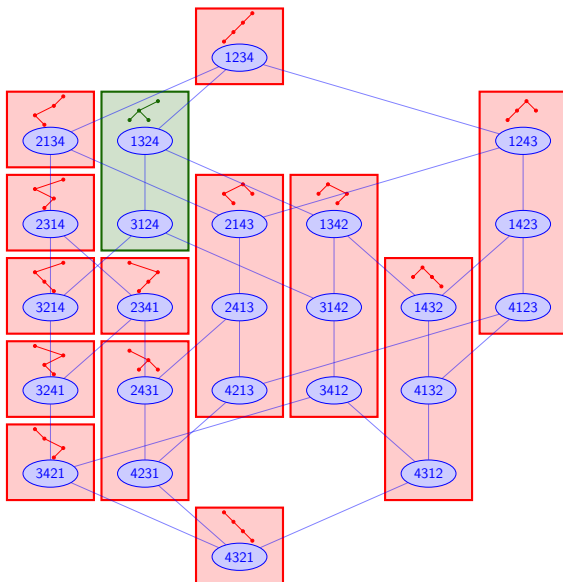


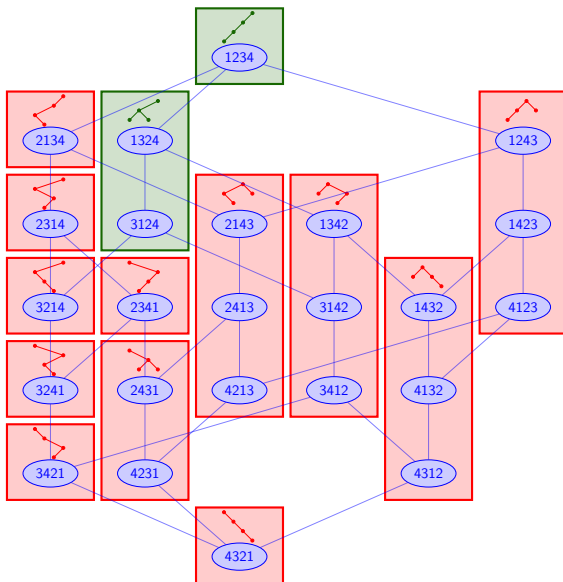
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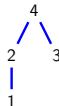
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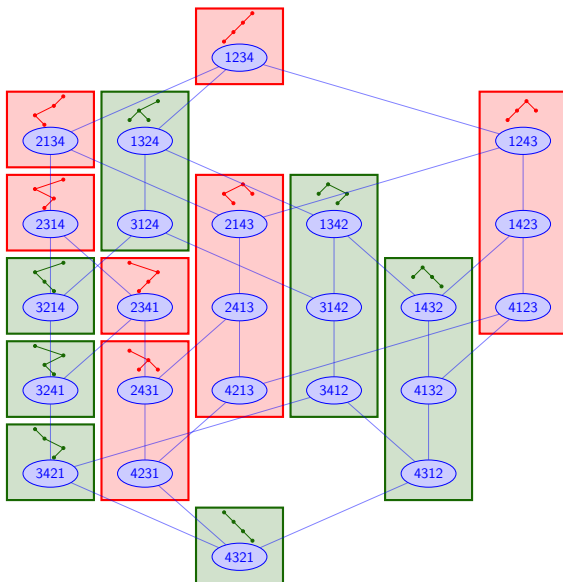






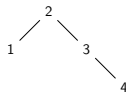
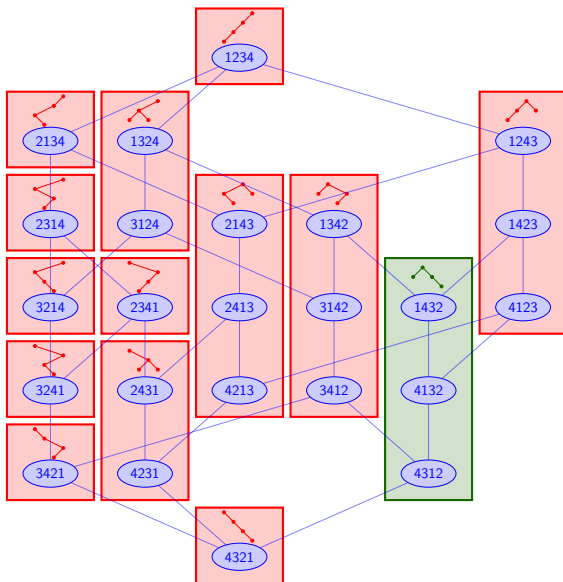
$F_{\leq}(T)$

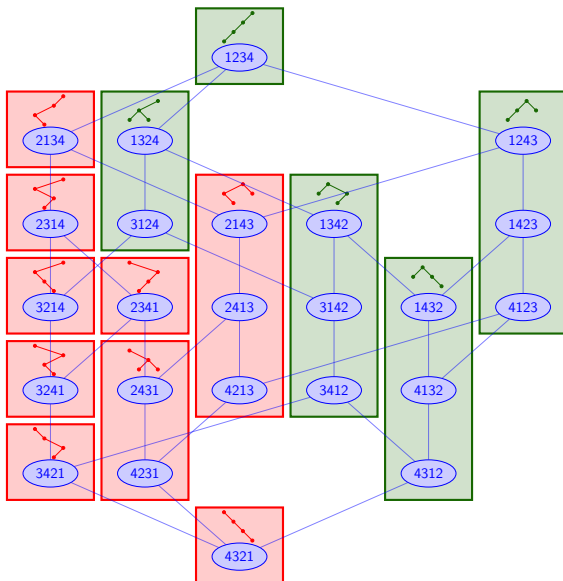




$$F_{\geq}(T)$$

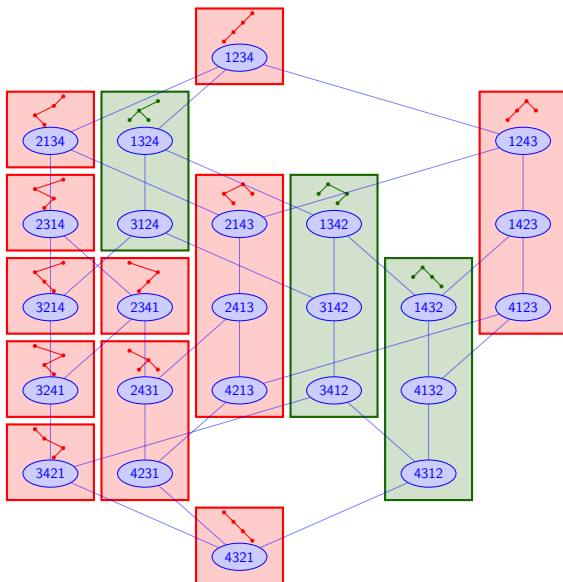






$$F_{\leq}(T')$$

2 3 4
|
1



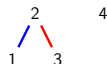
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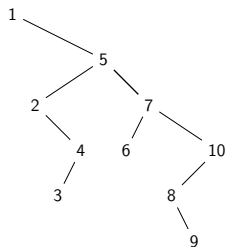
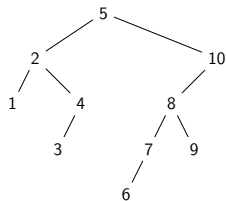


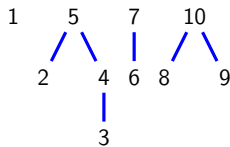
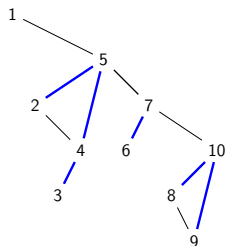
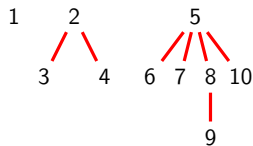
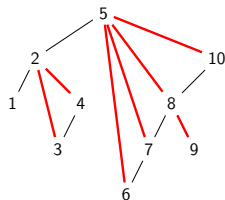
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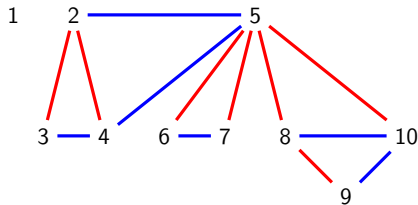
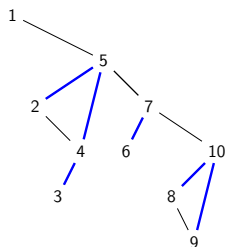
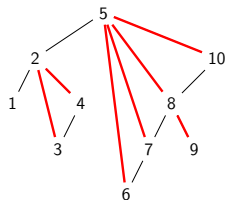


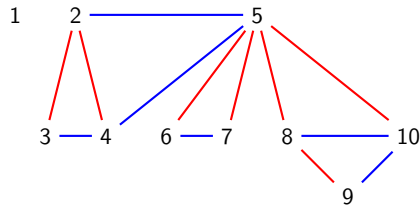
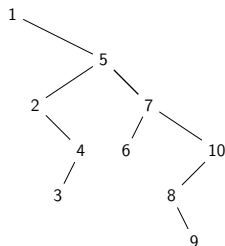
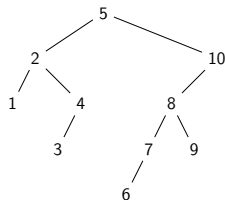
Intervalle-poset
[T, T']

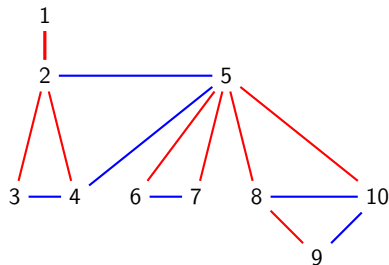
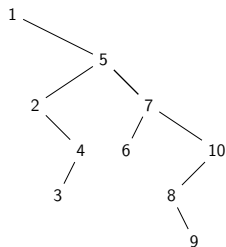
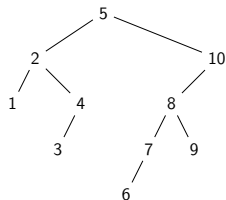


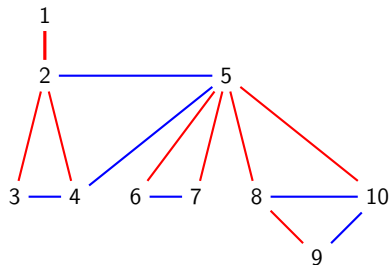
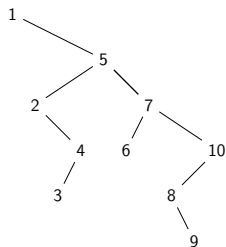
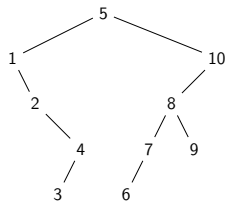


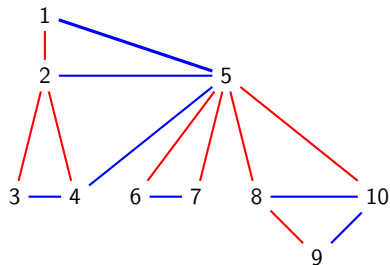
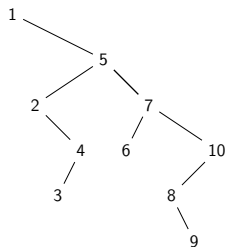
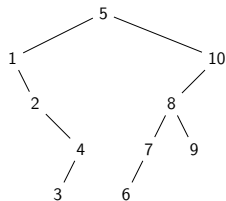


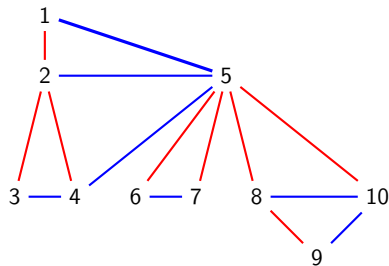
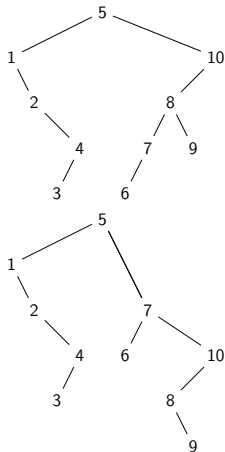


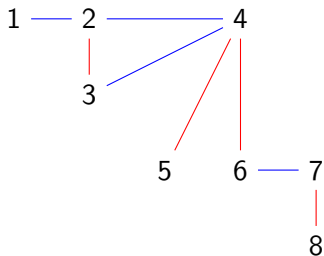


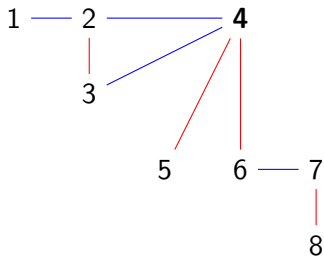


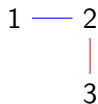
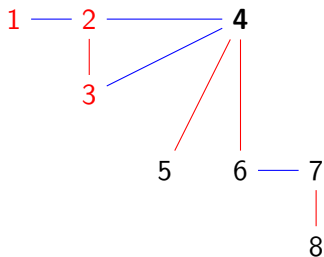


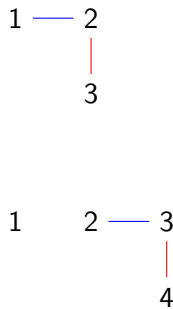
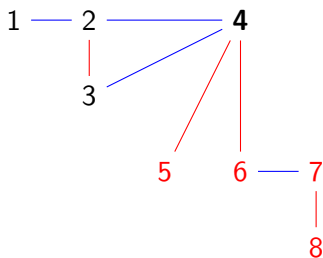


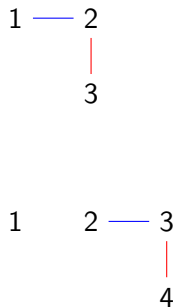
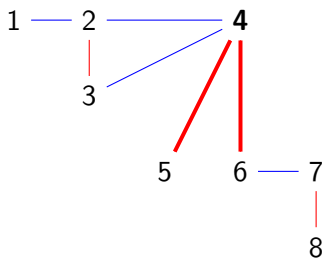




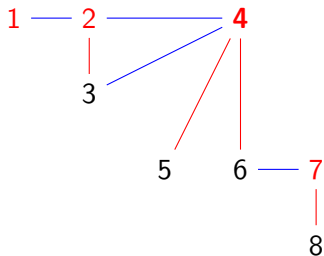




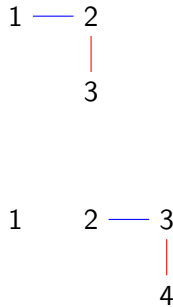




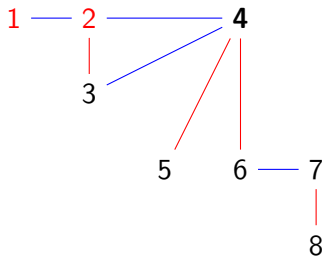
2



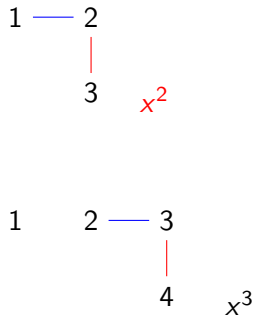
x^4



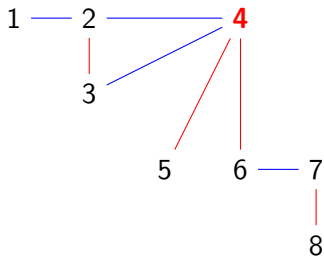
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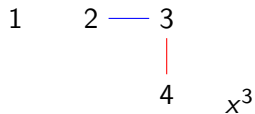
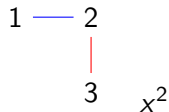
$$x^4 = x^2 \cdot x \cdot \frac{x^3}{x^2}$$



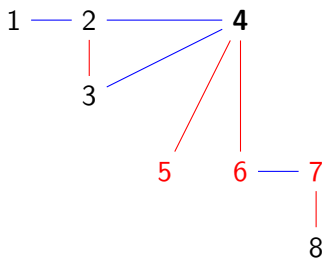
2



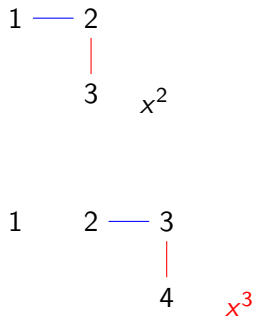
$$x^4 = x^2 \cdot \textcolor{red}{x} \cdot \frac{x^3}{x^2}$$



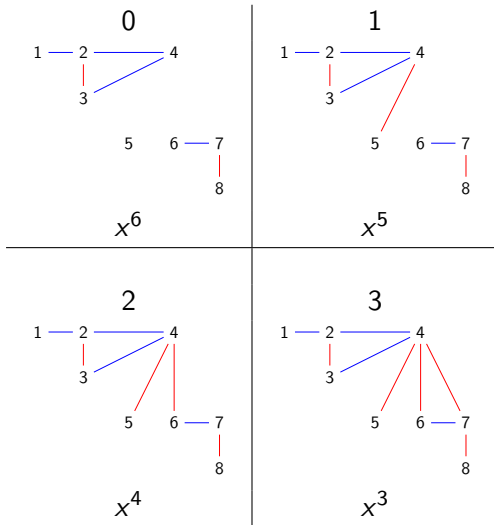
2



$$x^4 = x^2 \cdot x \cdot \frac{x^3}{x^2}$$



2



Theorem (Chapoton)

The generating functions of Tamari intervals satisfy the functional equation

$$\Phi(x, y) = B(\Phi, \Phi) + 1$$

where

$$B(f, g) = xyf(x, y) \frac{xg(x, y) - g(1, y)}{x - 1}$$

$$\begin{array}{c} 1 \text{ --- } 2 \\ | \\ 3 \end{array} = \left[\begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right]$$

$$\begin{array}{c} 1 \quad 2 \text{ --- } 3 \\ | \\ 4 \end{array} = \left[\begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right]$$

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x^2

$$\begin{array}{c} 1 \\ 2 \text{ --- } 3 \\ | \\ 4 \end{array} = \left[\begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right]$$

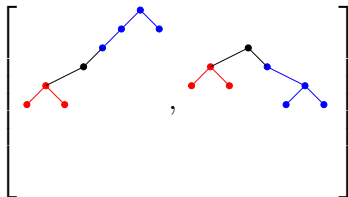
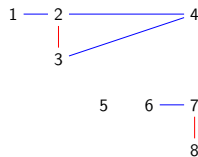
x^3

$$\begin{array}{c} 1 \text{ --- } 2 \\ | \\ 3 \end{array} = \left[\begin{array}{c} \bullet \text{ --- } \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \text{ --- } \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right]$$

x^2

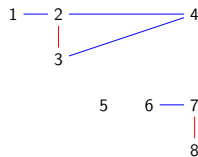
$$\begin{array}{c} 1 \quad 2 \text{ --- } 3 \\ | \\ 4 \end{array} = \left[\begin{array}{c} \bullet \text{ --- } \bullet \text{ --- } \bullet \\ / \quad \backslash \quad / \quad \backslash \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \text{ --- } \bullet \text{ --- } \bullet \\ / \quad \backslash \quad / \quad \backslash \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} \right]$$

x^3



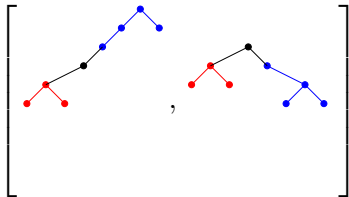
$$\begin{array}{c} 1 \text{ --- } 2 \\ | \\ 3 \end{array} = \left[\begin{array}{c} \bullet \text{ --- } \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \text{ --- } \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right]$$

x^2



$$\begin{array}{c} 1 \quad 2 \text{ --- } 3 \\ | \\ 4 \end{array} = \left[\begin{array}{c} \bullet \text{ --- } \bullet \text{ --- } \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \text{ --- } \bullet \text{ --- } \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right]$$

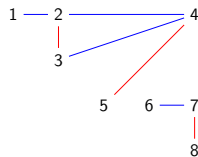
x^3



$$x^2 \cdot x \cdot x^3$$

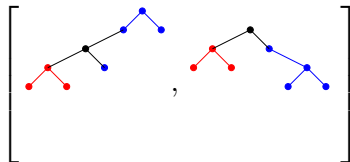
$$\begin{array}{c} 1 \text{ --- } 2 \\ | \\ 3 \end{array} = \left[\begin{array}{c} \bullet \text{ --- } \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \text{ --- } \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} \right]$$

x^2



$$\begin{array}{c} 1 \quad 2 \text{ --- } 3 \\ | \\ 4 \end{array} = \left[\begin{array}{c} \bullet \text{ --- } \bullet \text{ --- } \bullet \\ | \quad | \quad | \\ \bullet \quad \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \text{ --- } \bullet \text{ --- } \bullet \\ | \quad | \quad | \\ \bullet \quad \bullet \quad \bullet \end{array} \right]$$

x^3



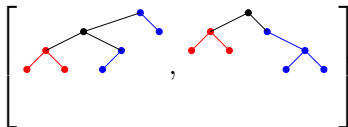
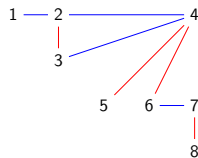
$$x^2 \cdot x \cdot x^3 + x^2 \cdot x \cdot x^2$$

$$\begin{array}{c} 1 \text{ --- } 2 \\ | \\ 3 \end{array} = \left[\begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right]$$

x^2

$$\begin{array}{c} 1 \quad 2 \text{ --- } 3 \\ | \\ 4 \end{array} = \left[\begin{array}{c} \bullet \quad \bullet \quad \bullet \\ / \quad \backslash \quad / \quad \backslash \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right]$$

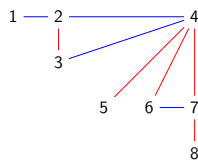
x^3



$$x^2 \cdot x \cdot x^3 + x^2 \cdot x \cdot x^2 + x^2 \cdot x \cdot x$$

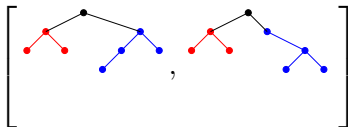
$$\begin{array}{c} 1 \text{ --- } 2 \\ | \\ 3 \end{array} = \left[\begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right]$$

x^2



$$\begin{array}{c} 1 \quad 2 \text{ --- } 3 \\ | \\ 4 \end{array} = \left[\begin{array}{c} \bullet \quad \bullet \quad \bullet \\ / \quad \backslash \quad / \quad \backslash \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ / \quad \backslash \quad / \quad \backslash \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} \right]$$

x^3



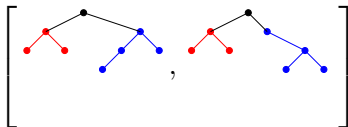
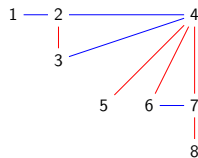
$$x^2 \cdot x \cdot x^3 + x^2 \cdot x \cdot x^2 + x^2 \cdot x \cdot x + x^2 \cdot x$$

$$\begin{array}{c} 1 \text{ --- } 2 \\ | \\ 3 \end{array} = \left[\begin{array}{c} \bullet \text{ --- } \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \text{ --- } \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right]$$

x^2

$$\begin{array}{c} 1 \quad 2 \text{ --- } 3 \\ | \\ 4 \end{array} = \left[\begin{array}{c} \bullet \text{ --- } \bullet \text{ --- } \bullet \\ / \quad \backslash \quad / \quad \backslash \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \text{ --- } \bullet \text{ --- } \bullet \\ / \quad \backslash \quad / \quad \backslash \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} \right]$$

x^3



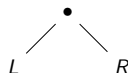
$$x^2 \cdot x \cdot (x^3 + x^2 + x + 1)$$

Tamari Polynomials

\mathcal{B}_T is recursively defined by

$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x) \frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$

avec $T =$ 

Theorem (Châtel, P.)


\mathcal{B}_T counts the number of trees smaller than or equal to T in the Tamari lattice according to the number of nodes on their left border.

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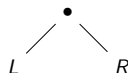
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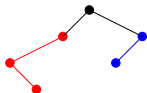
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$$\mathcal{B}_L(x) = x^3 + x^2$$



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$$\mathcal{B}_R(x) = x^2$$



$$\mathcal{B}_{\emptyset} := 1$$

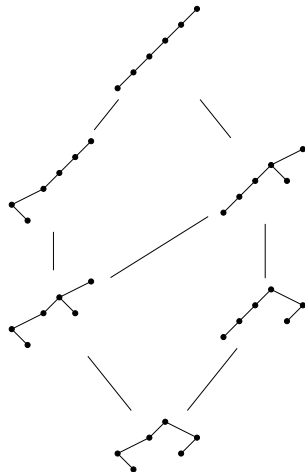
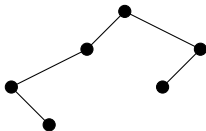
$$\mathcal{B}_T(x) := x(\textcolor{red}{x^3} + \textcolor{red}{x^2}) \frac{\textcolor{blue}{x}\mathcal{B}_R(x) - \mathcal{B}_R(1)}{\textcolor{blue}{x} - 1}$$

$$\textcolor{blue}{\mathcal{B}_R(x)} = \textcolor{blue}{x^2}$$

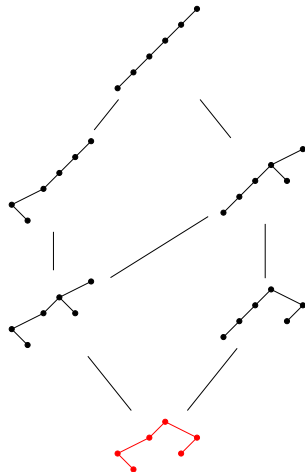
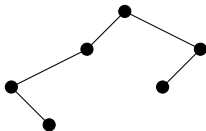


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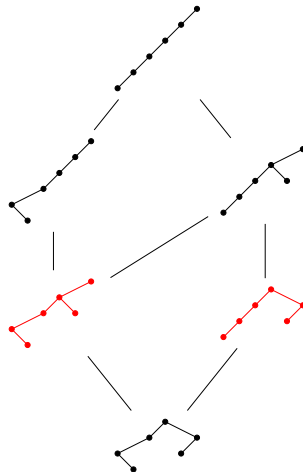
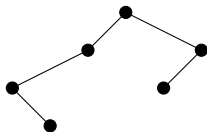
$$\mathcal{B}_T(x) := x(\textcolor{red}{x}^3 + \textcolor{red}{x}^2)(1 + \textcolor{blue}{x} + \textcolor{blue}{x}^2)$$



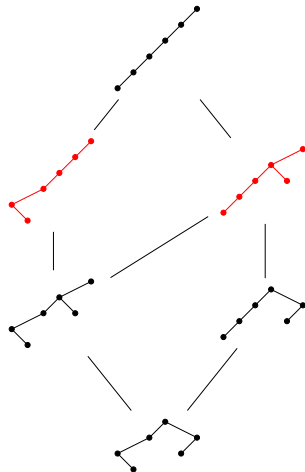
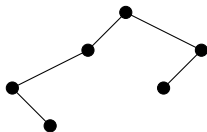
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



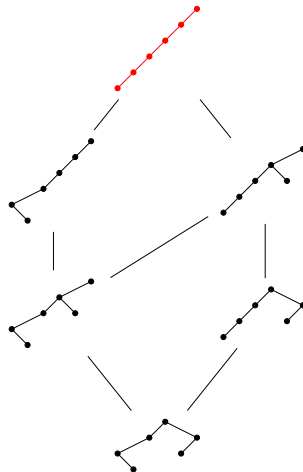
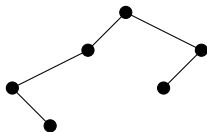
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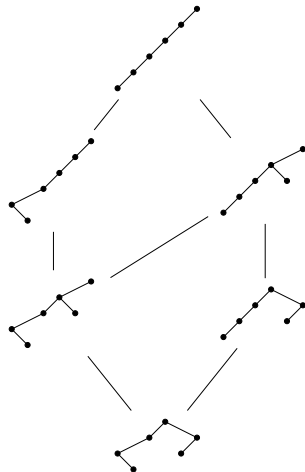
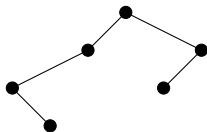
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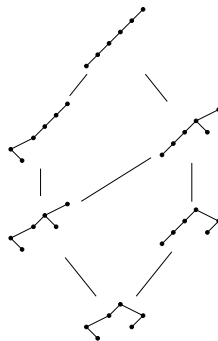
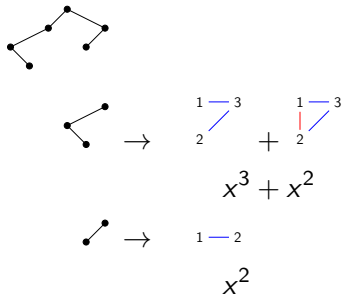


$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$

$$\mathcal{B}_T(1) = 6$$





$$\begin{array}{c} 1 \text{ --- } 3 \\ \quad \diagdown \\ 2 \end{array} + \begin{array}{c} 1 \text{ --- } 3 \\ \quad \diagup \\ 2 \end{array}$$

$$x^3 + x^2$$

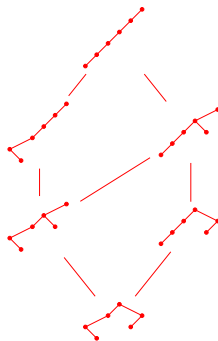


$$1 \text{ --- } 2$$

$$x^2$$

$$\begin{array}{c} 1 \text{ --- } 3 \text{ --- } 4 \\ \quad \diagdown \\ 2 \quad 5 \text{ --- } 6 \end{array}$$

$$x^3 \cdot x \cdot x^2$$





→

$$\begin{array}{c} 1 \text{ --- } 3 \\ \quad \diagup \\ 2 \end{array} + \begin{array}{c} 1 \text{ --- } 3 \\ \quad \diagdown \\ 2 \end{array}$$

$$x^3 + x^2$$



→

$$1 \text{ --- } 2$$

$$x^2$$

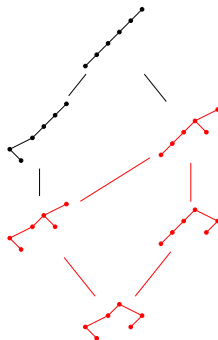
$$\begin{array}{c} 1 \text{ --- } 3 \text{ --- } 4 \\ \quad \diagup \\ 2 \quad 5 \text{ --- } 6 \end{array}$$

+

$$\begin{array}{c} 1 \text{ --- } 3 \text{ --- } 4 \\ \quad \diagup \\ 2 \quad 5 \text{ --- } 6 \end{array}$$

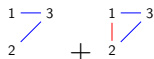
$$x^3 \cdot x \cdot x^2$$

$$+ x^3 \cdot x \cdot x$$





→



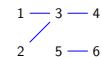
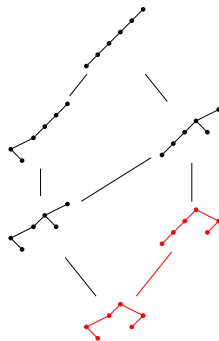
$$x^3 + x^2$$



→



$$x^2$$



+



+



$$x^3 \cdot x \cdot x^2$$

$$+ x^3 \cdot x \cdot x + x^3 \cdot x$$



→

$$\begin{array}{c} 1 \text{ --- } 3 \\ | \quad \diagup \\ 2 \end{array} + \begin{array}{c} 1 \text{ --- } 3 \\ | \quad \diagup \\ 2 \end{array}$$

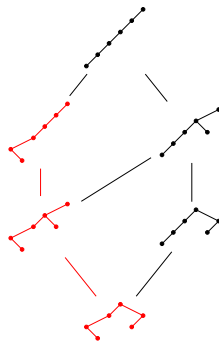
$$x^3 + x^2$$



→

$$1 \text{ --- } 2$$

$$x^2$$



$$\begin{array}{c} 1 \text{ --- } 3 \text{ --- } 4 \\ | \quad \diagup \quad | \\ 2 \quad 5 \text{ --- } 6 \end{array}$$

$$+ \begin{array}{c} 1 \text{ --- } 3 \text{ --- } 4 \\ | \quad \diagup \quad | \\ 2 \quad 5 \text{ --- } 6 \end{array}$$

$$+ \begin{array}{c} 1 \text{ --- } 3 \text{ --- } 4 \\ | \quad \diagup \quad | \\ 2 \quad 5 \text{ --- } 6 \end{array}$$

$$+ \begin{array}{c} 1 \text{ --- } 3 \text{ --- } 4 \\ | \quad \diagup \quad | \\ 2 \quad 5 \text{ --- } 6 \end{array}$$

$$x^3 \cdot x \cdot x^2$$

$$+ x^3 \cdot x \cdot x + x^3 \cdot x$$

$$+ x^2 \cdot x \cdot x^2$$



→

$$\begin{array}{c} 1 \text{ --- } 3 \\ \quad \diagup \\ 2 \end{array} + \begin{array}{c} 1 \text{ --- } 3 \\ \quad \textcolor{red}{\diagup} \\ 2 \end{array}$$

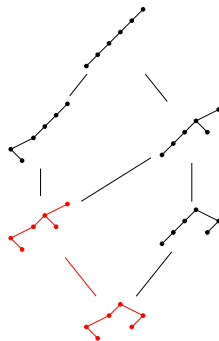
$$x^3 + \textcolor{red}{x}^2$$



→

$$1 \text{ --- } 2$$

$$\textcolor{red}{x}^2$$



$$\begin{array}{c} 1 \text{ --- } 3 \text{ --- } 4 \\ \quad \diagup \\ 2 \quad 5 \text{ --- } 6 \end{array}$$

$$+ \begin{array}{c} 1 \text{ --- } 3 \text{ --- } 4 \\ \quad \diagup \quad \textcolor{red}{\diagup} \\ 2 \quad 5 \text{ --- } 6 \end{array}$$

$$+ \begin{array}{c} 1 \text{ --- } 3 \text{ --- } 4 \\ \quad \diagup \quad \textcolor{red}{\diagup} \quad \textcolor{red}{\diagup} \\ 2 \quad 5 \text{ --- } 6 \end{array}$$

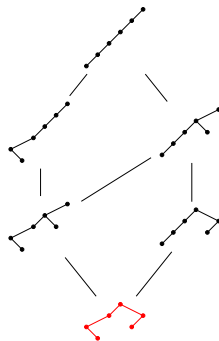
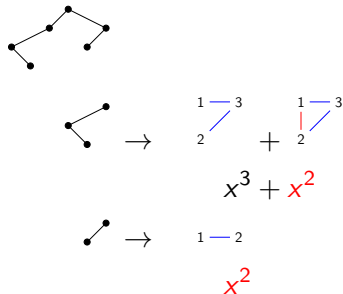
$$x^3 \cdot x \cdot x^2$$

$$+ x^3 \cdot x \cdot x + x^3 \cdot x$$

$$+ \begin{array}{c} 1 \text{ --- } 3 \text{ --- } 4 \\ \quad \textcolor{red}{\diagup} \quad \diagup \\ 2 \quad 5 \text{ --- } 6 \end{array}$$

$$+ \begin{array}{c} 1 \text{ --- } 3 \text{ --- } 4 \\ \quad \textcolor{red}{\diagup} \quad \textcolor{red}{\diagup} \quad \diagup \\ 2 \quad 5 \text{ --- } 6 \end{array}$$

$$+ x^2 \cdot x \cdot x^2 + \textcolor{red}{x}^2 \cdot x \cdot x$$



$$\begin{array}{ccccccc}
 \begin{array}{c} 1 \text{---} 3 \text{---} 4 \\ | \quad \diagup \quad \text{---} \\ 2 \quad 5 \text{---} 6 \end{array} & + & \begin{array}{c} 1 \text{---} 3 \text{---} 4 \\ | \quad \diagup \quad \text{---} \\ 2 \quad 5 \text{---} 6 \end{array} & + & \begin{array}{c} 1 \text{---} 3 \text{---} 4 \\ | \quad \diagup \quad \text{---} \\ 2 \quad 5 \text{---} 6 \end{array} & + & \begin{array}{c} 1 \text{---} 3 \text{---} 4 \\ | \quad \diagup \quad \text{---} \\ 2 \quad 5 \text{---} 6 \end{array} & + & \begin{array}{c} 1 \text{---} 3 \text{---} 4 \\ | \quad \diagup \quad \text{---} \\ 2 \quad 5 \text{---} 6 \end{array} \\
 x^3 \cdot x \cdot x^2 & & + x^3 \cdot x \cdot x & + & x^3 \cdot x & & + x^2 \cdot x \cdot x^2 & + & x^2 \cdot x \cdot x & + & x^2 \cdot x
 \end{array}$$



→

$$\begin{array}{c} 1 \text{ --- } 3 \\ \quad \diagup \\ 2 \end{array} + \begin{array}{c} 1 \text{ --- } 3 \\ \quad \diagup \\ 2 \end{array}$$

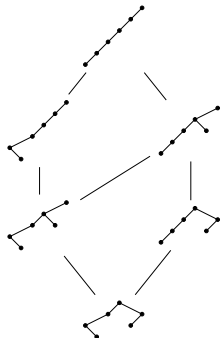
$$x^3 + x^2$$



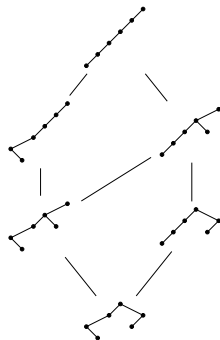
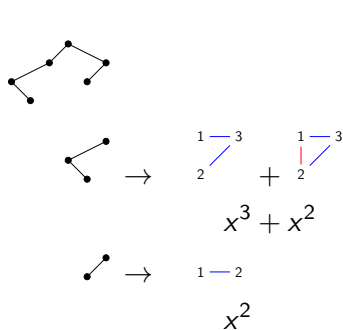
→

$$1 \text{ --- } 2$$

$$x^2$$



$$(x^3 + x^2).x.(x^2 + x + 1) =$$



$$(x^3 + x^2).x.(x^2 + x + 1) = x \mathcal{B}_L(x) \frac{\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$