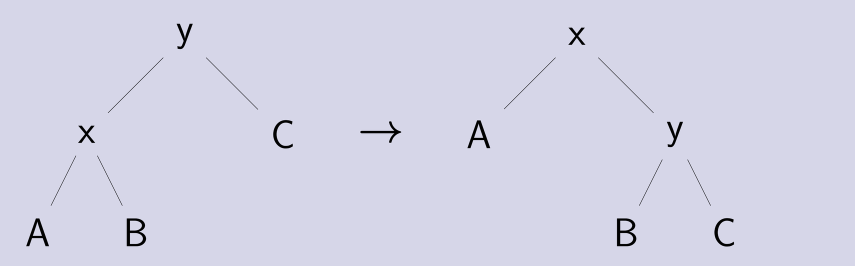
Two bijections on Tamari intervals

F. Chapoton, G. Chatel, V. Pons

Université Claude Bernard Lyon
Université Paris-Est Marne-la-Vallée
Fakultät für Mathematik, Universität Wien

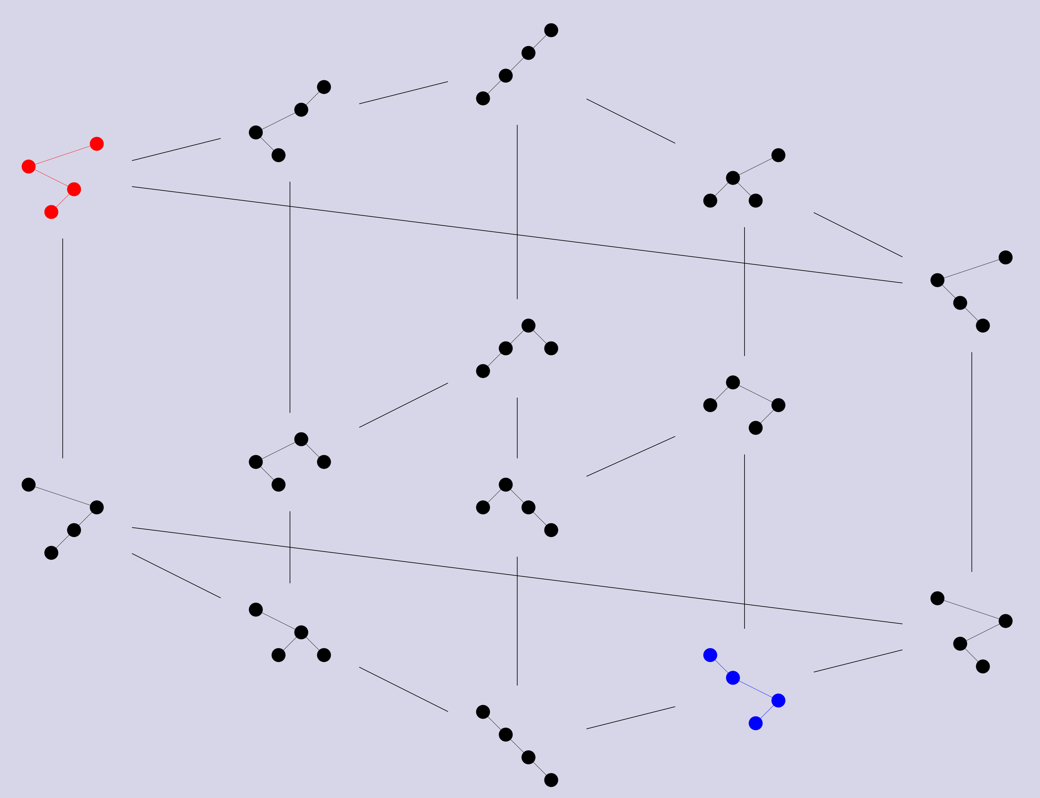
Definitions

The right rotation on binary trees

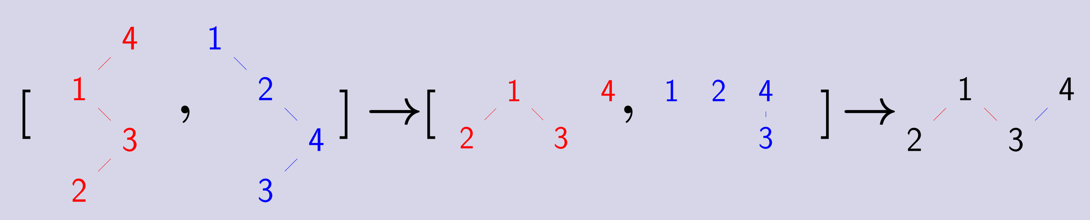


The right rotation is the cover relation of the Tamari order on binary trees.

The Tamari lattice



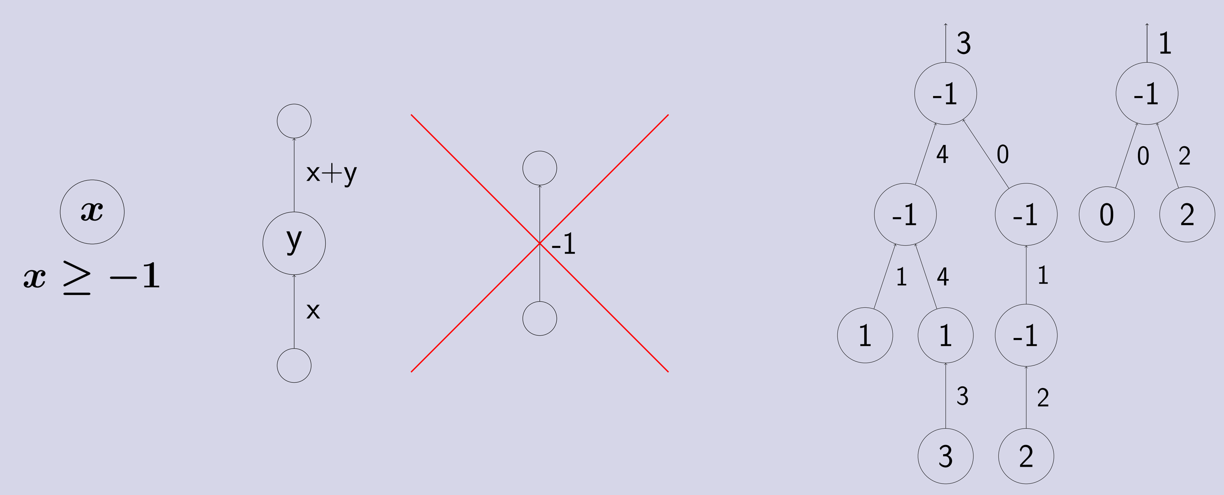
Interval-poset



Rules of interval-posets:

- $a < c$ and $a \lhd c \implies b \lhd c, b \in [a,c]$.
- $a < c$ and $c \lhd a \implies b \lhd a, b \in [a,c]$.

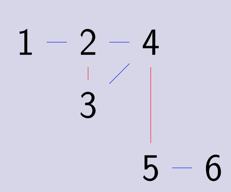
Flows of forest of rooted trees

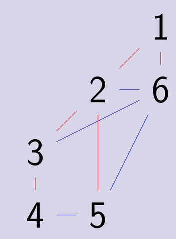


The *exit rate* of a forest is the sum of the exit rates of the trees.

A *closed flow* is a flow of a forest of rooted trees with exit rate 0.

Statistics on interval-posets

$I =$  $trees(I) = 4$, $ir(I) = 1$

$J =$  $trees(J) = 1$, $ir(J) = 4$

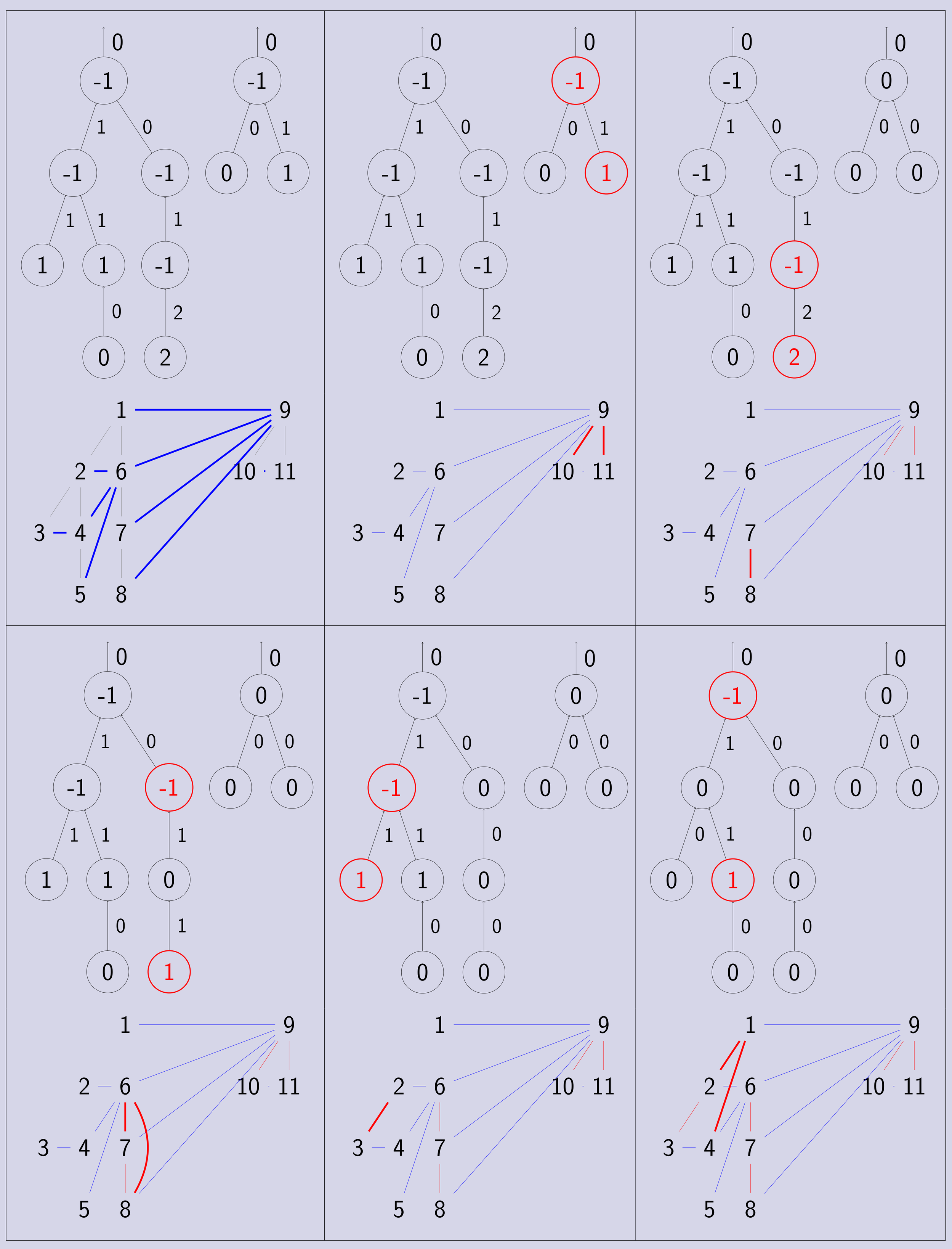
$trees(I)$ = number of red trees of I, $ir(I)$ = largest k s.t. there is no relation $i \lhd i+1$ for $i \in [1,k]$.

Results

Theorem

The number of closed flows of a given forest F is the number of elements smaller than or equal to a certain binary tree $T(F)$ in the Tamari order.

Bijective proof



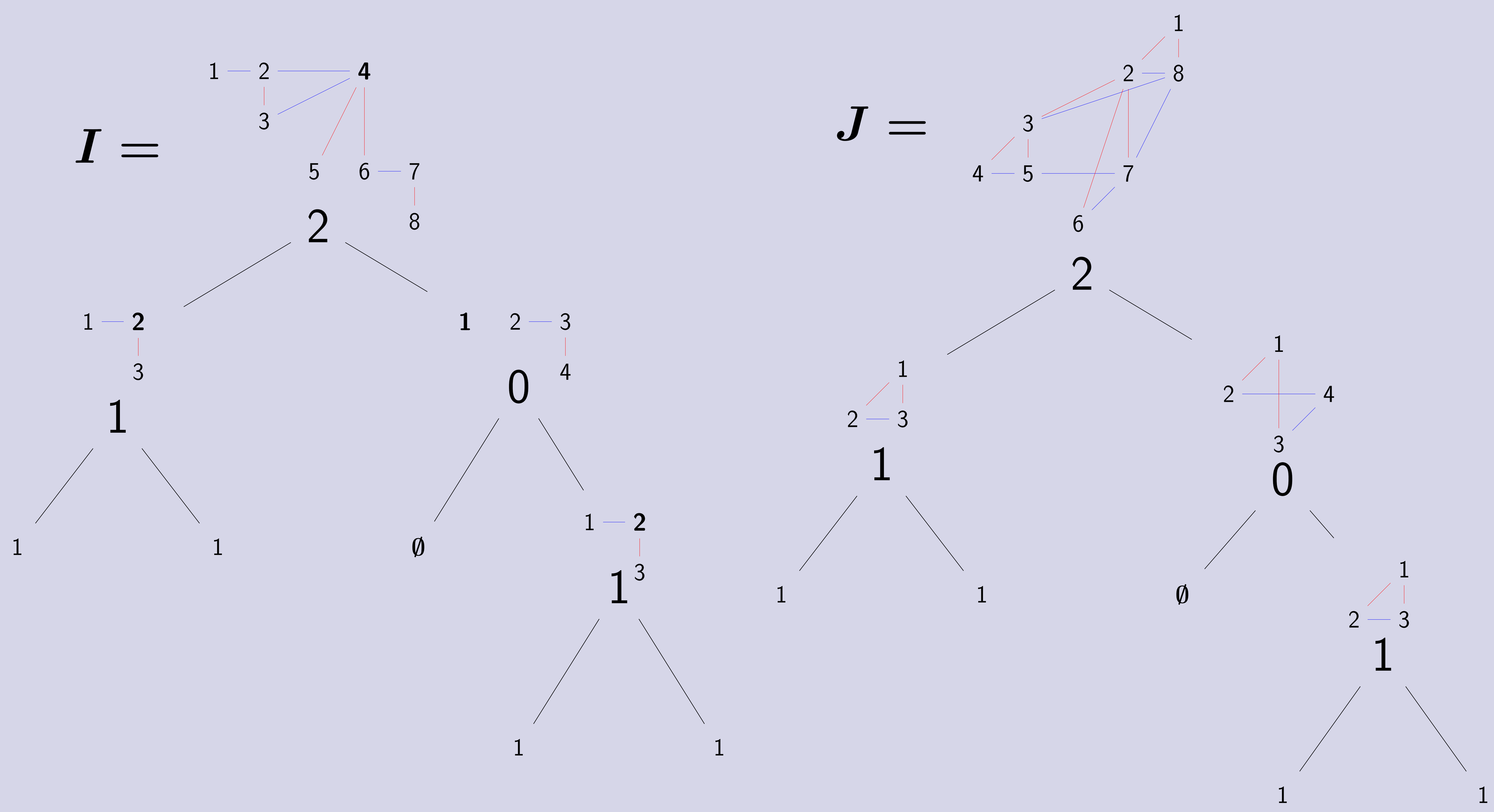
Comments

- How statistics on flows can be read on the corresponding interval-poset ?
- The flows appear in the study of the pre-Lie operad (see [2]) which doesn't seem related to the Tamari order. Can we provide an explanation of this link ?
- Each open flow can be sent to a unique closed flow. What is the connection with the Tamari order ?

Theorem

Let I be an interval-poset of size n such that $trees(I) = x$ and $ir(I) = y$. There exists another interval-poset J of size n such that $trees(J) = y$ and $ir(J) = x$.

Bijective proof



$$\begin{aligned} trees(I) &= 4 = ir(J) \\ ir(I) &= 1 = trees(J) \end{aligned}$$

Comments

This bijection yields a non trivial equality between two functionnal equations. Can we prove it algebrically ?

Citations

M. Bousquet-Mélou, E. Fusy, and L.-F. Préville-Ratelle.
The number of intervals in the m-Tamari lattices.
Electron. J. Combin., 18(2):Paper 31, 26, 2011.

F. Chapoton.
Flows on rooted trees and the Menous-Novelli-Thibon idempotents.
arXiv:1203.1780, 2013.