Intervals of the Tamari lattice

Viviane Pons

Universität Wien

LRI, 28 mars 2014

Ordre faible

251436

Ordre faible

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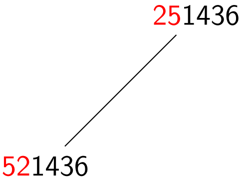
Ordre faible



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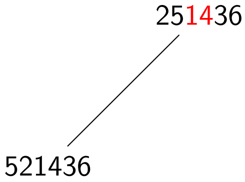

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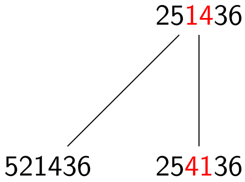

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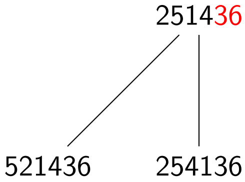

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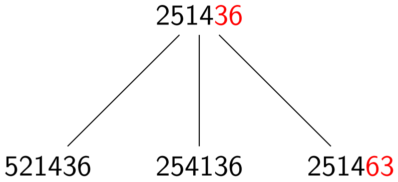

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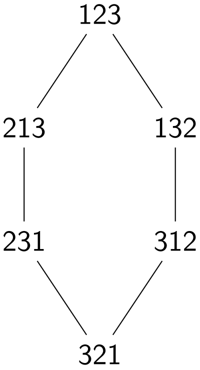

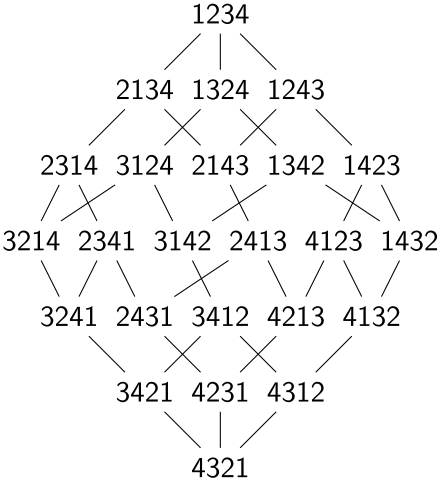

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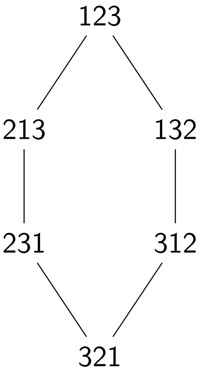

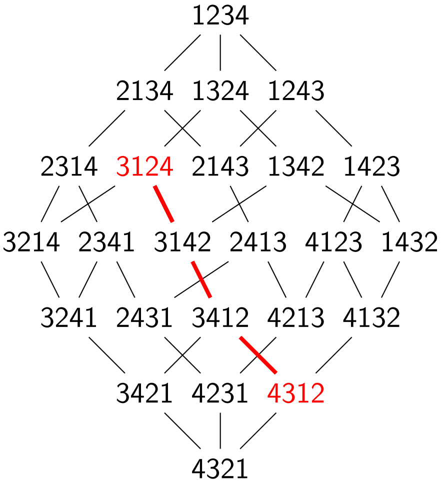

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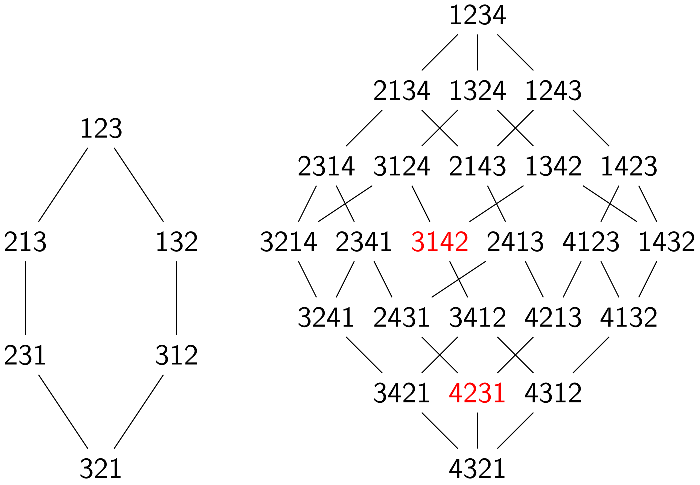

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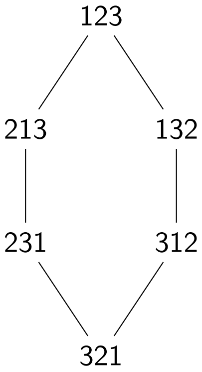

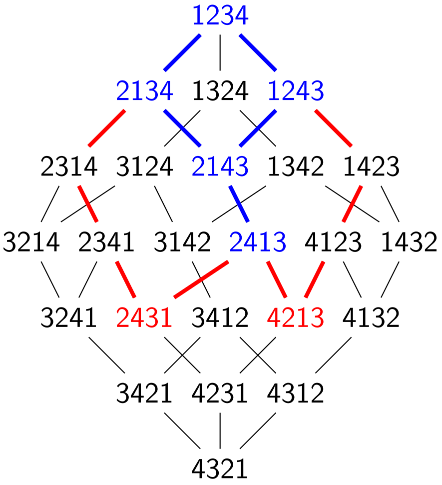

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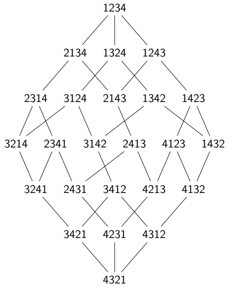
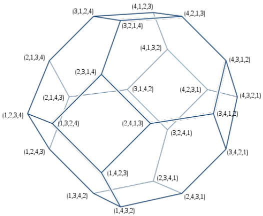
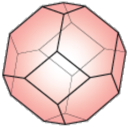






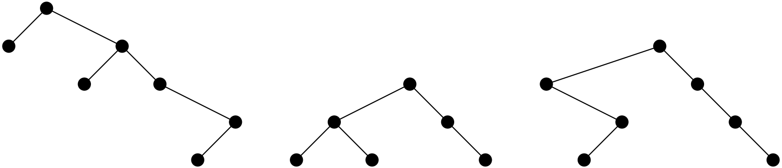
Treillis de Tamari

- 1962, Tamari : ordre sur les parenthésages formels
- 1972, Huang, Tamari : structure de treillis

Treillis de Tamari

- 1962, Tamari : ordre sur les parenthésages formels
- 1972, Huang, Tamari : structure de treillis

Arbres binaires



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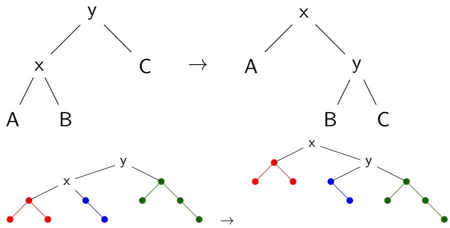

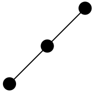

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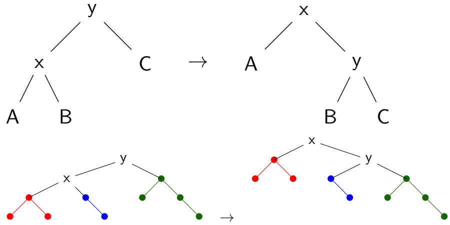
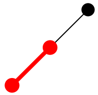
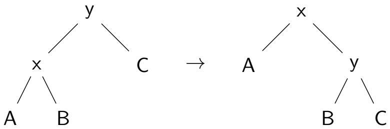
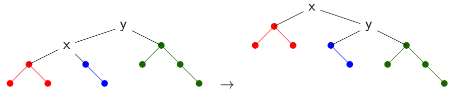
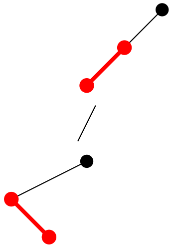




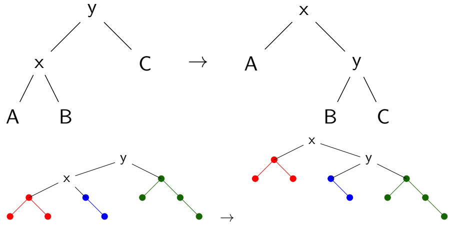
Rotation droite

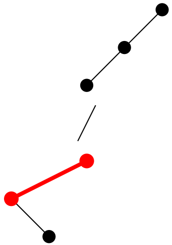






Rotation droite





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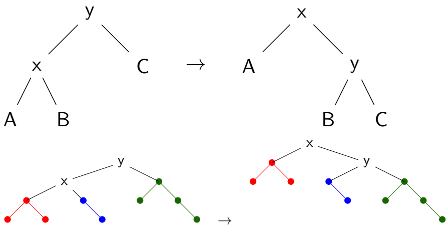
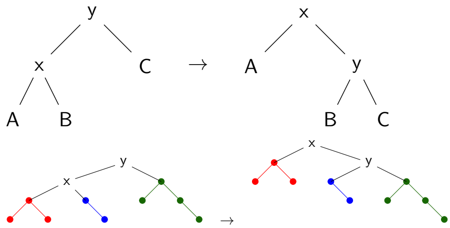
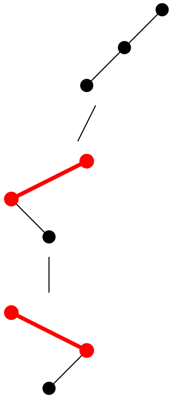
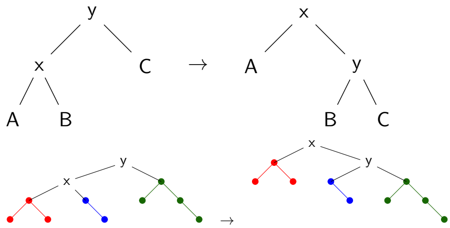


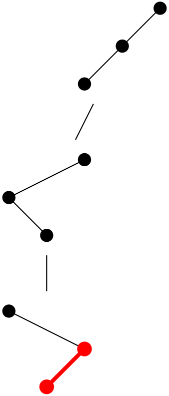
Rotation droite





Rotation droite





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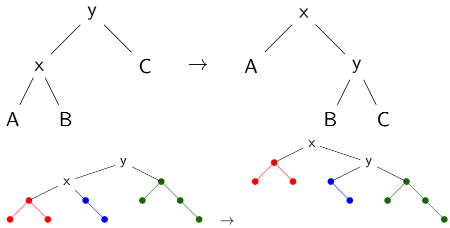

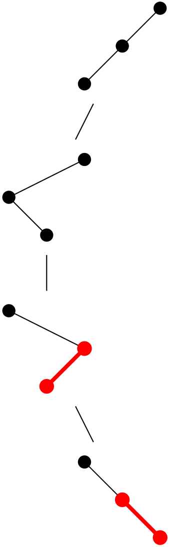

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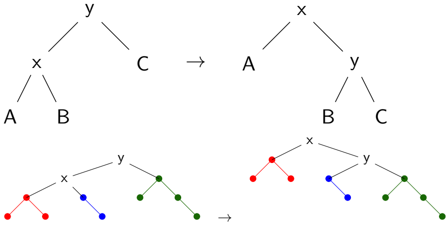

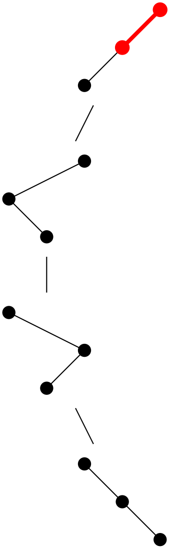

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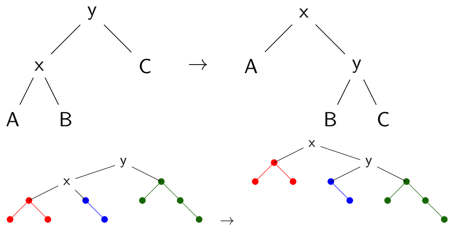

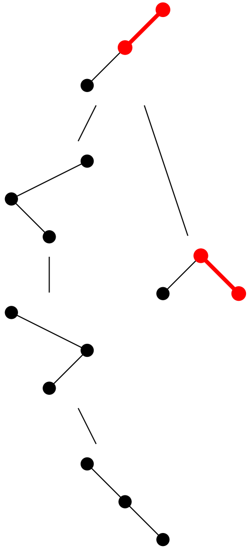

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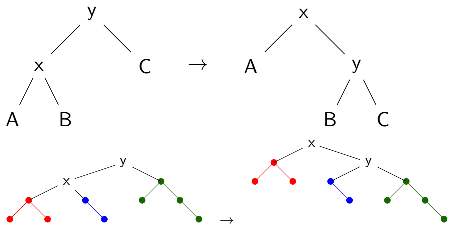
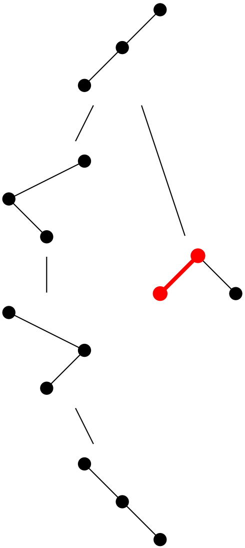
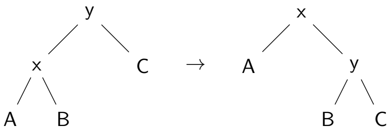
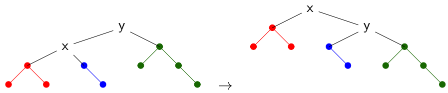
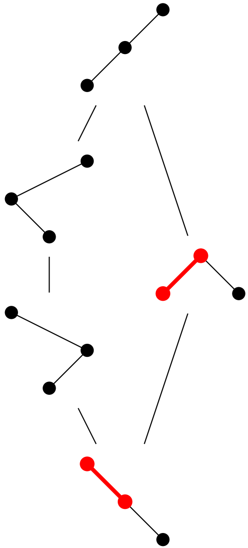




Rotation droite







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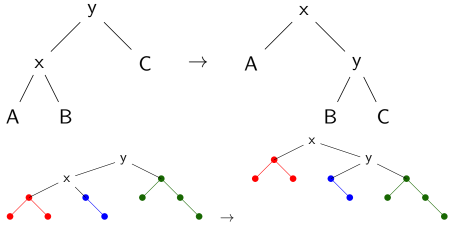

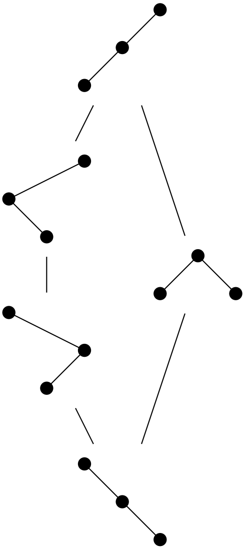

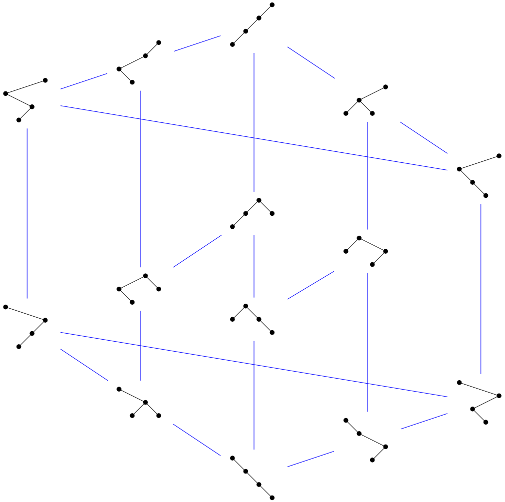

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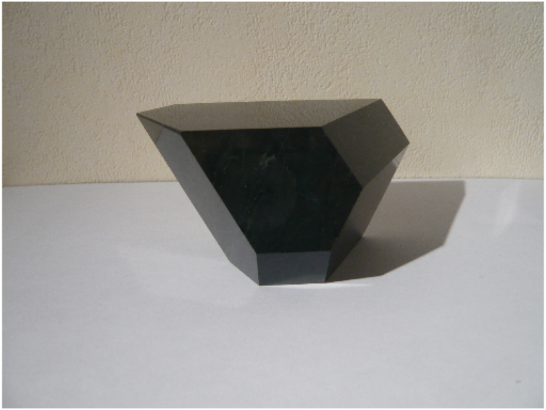
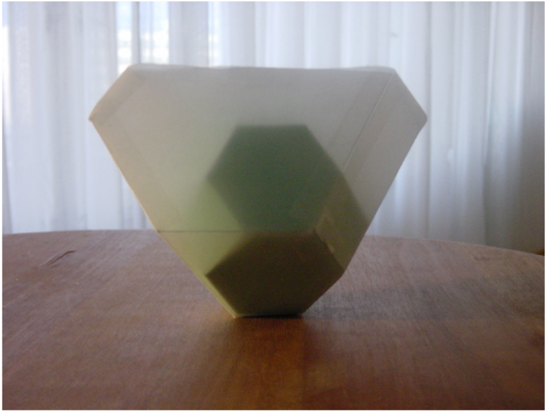
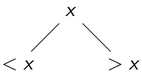


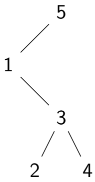
Associaèdre et permutoèdre

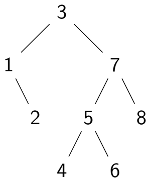


Lien avec l'ordre faible

Étiquetage canonique





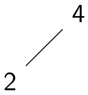


Insertion dans un arbre binaire de recherche

$$15324 \quad \rightarrow \quad 4$$

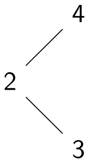
Insertion dans un arbre binaire de recherche

15324 $\rightarrow$



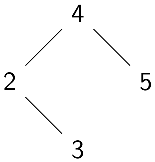
Insertion dans un arbre binaire de recherche

15324 $\rightarrow$



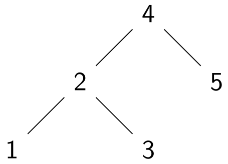
Insertion dans un arbre binaire de recherche

15324 $\rightarrow$



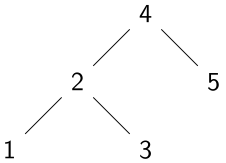
Insertion dans un arbre binaire de recherche

15324 $\rightarrow$



Insertion dans un arbre binaire de recherche

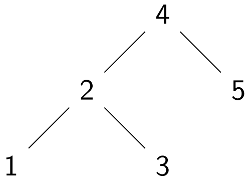
15324 $\rightarrow$

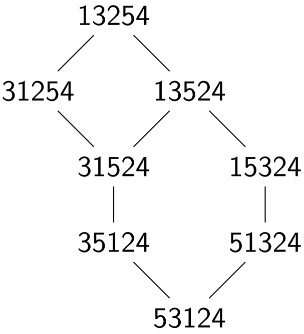


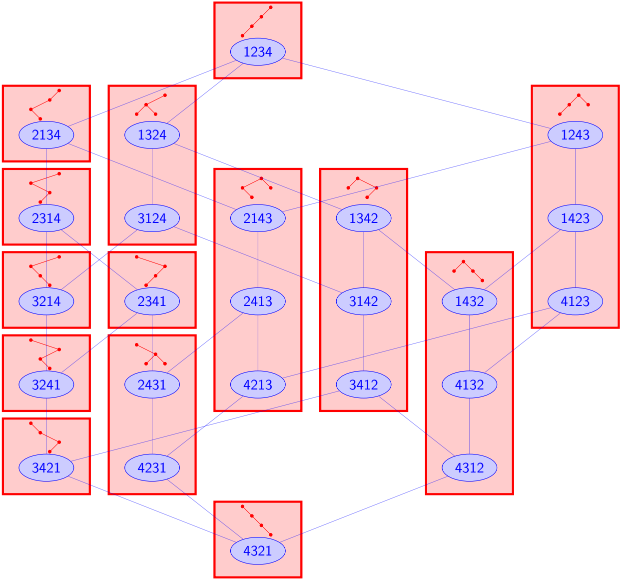
Caractérisation : les permutations qui correspondent à un arbre donné sont ses extensions linéaires

$15324, 31254, 35124, 51324, \ldots$

Insertion dans un arbre binaire de recherche






1234
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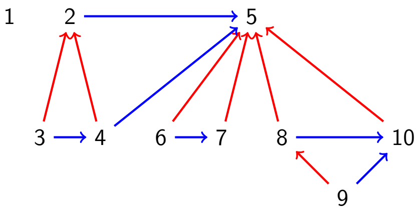
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321

Intervalles du treillis de Tamari

Chapoton 2007 : énumération des intervalles.

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$$

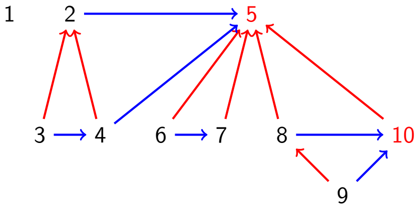


Definition

Un intervalle-poset est un poset de taille n étiqueté par $1, \dots, n$ *tel que*

- *Si* $a < c$ *et* $a \lhd c$ *alors* $b \lhd c$ *pour tout* $a < b < c$.
- *Si* $a < c$ *et* $c \lhd a$ *alors* $b \lhd a$ *pour tout* $a < b < c$.

On note $a \lhd b$ *pour a inférieur à b dans le poset.*

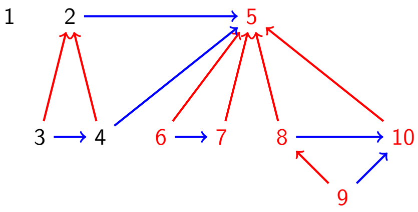


Definition

Un intervalle-poset est un poset de taille n étiqueté par $1, \dots, n$ tel que

- *Si $a < c$ et $a \lhd c$ alors $b \lhd c$ pour tout $a < b < c$.*
- *Si $a < c$ et $c \lhd a$ alors $b \lhd a$ pour tout $a < b < c$.*

On note $a \lhd b$ pour a inférieur à b dans le poset.



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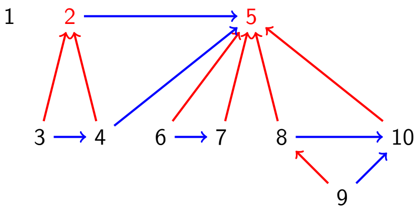

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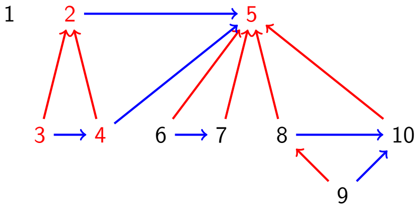

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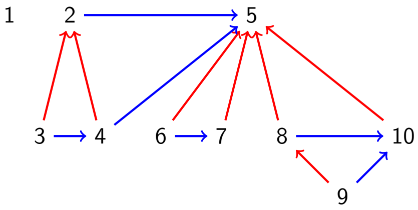
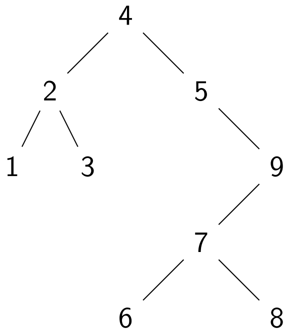


Theorem (Châtel, P.)

Les intervalles-posets sont en bijection avec les intervalles du treillis de Tamari.


4
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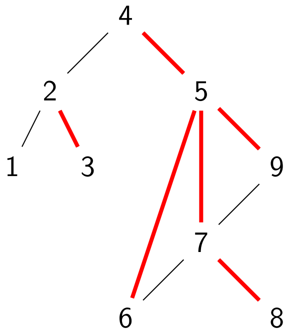
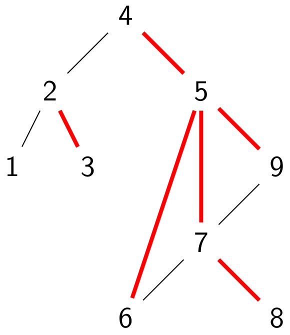
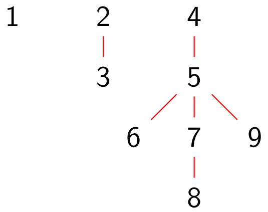
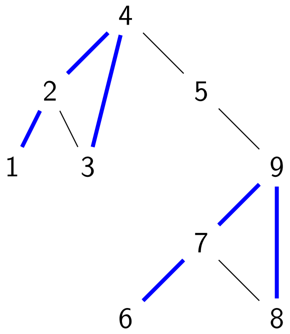
final forest $F_{\geq}(T)$



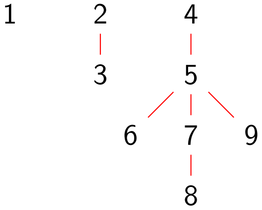


final forest $F_{\geq}(T)$

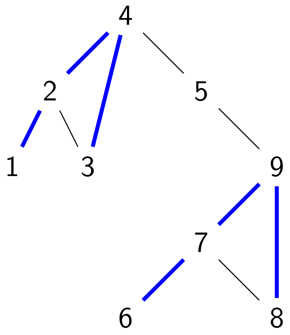




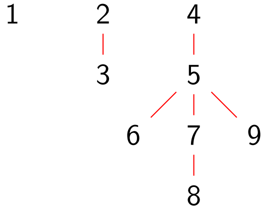
final forest $F_{\geq}(T)$



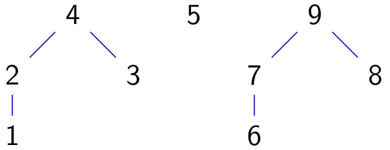
Initial forest $F_{\leq}(T)$

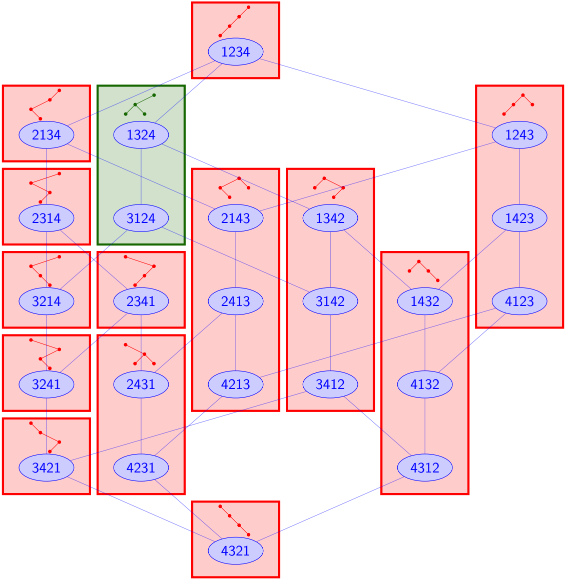


final forest $F_{\geq}(T)$



Initial forest $F_{\leq}(T)$

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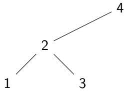
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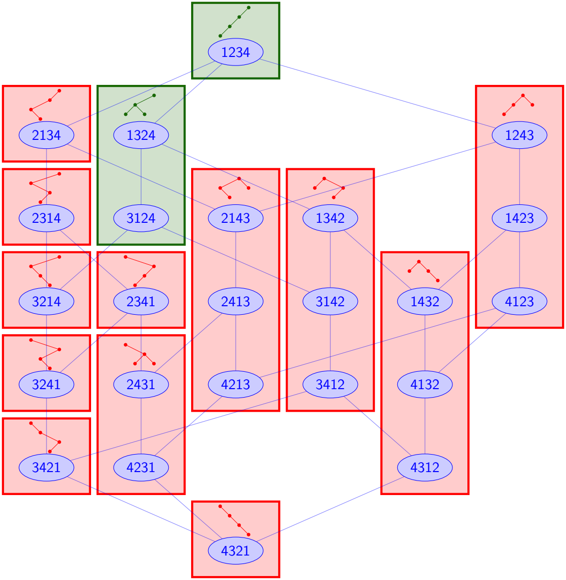
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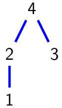
$F_{\leq}(T)$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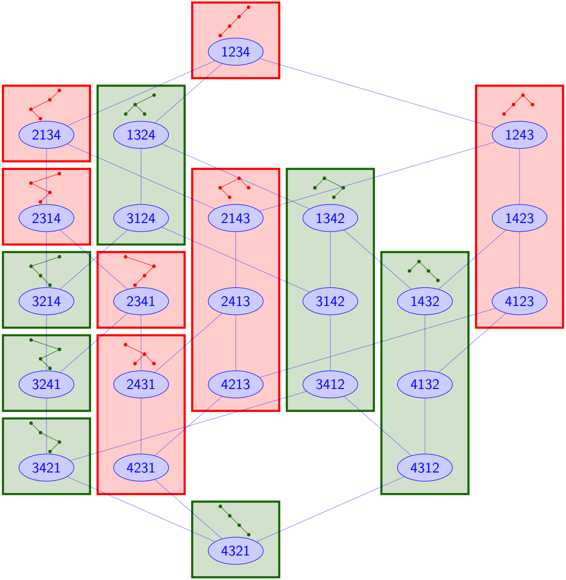
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$F_{\geq}(T)$


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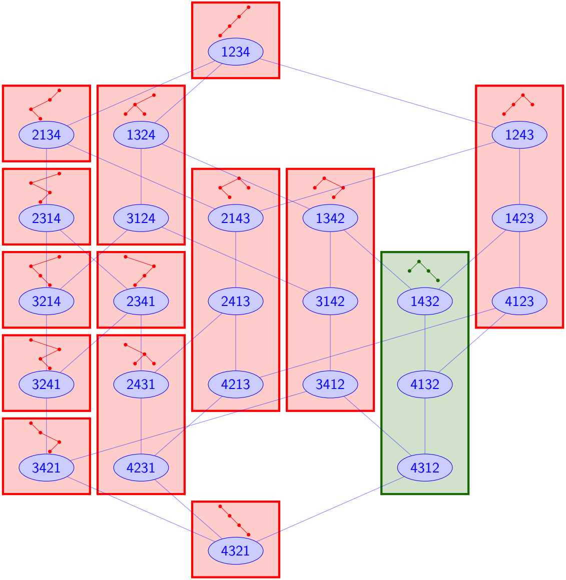
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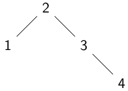
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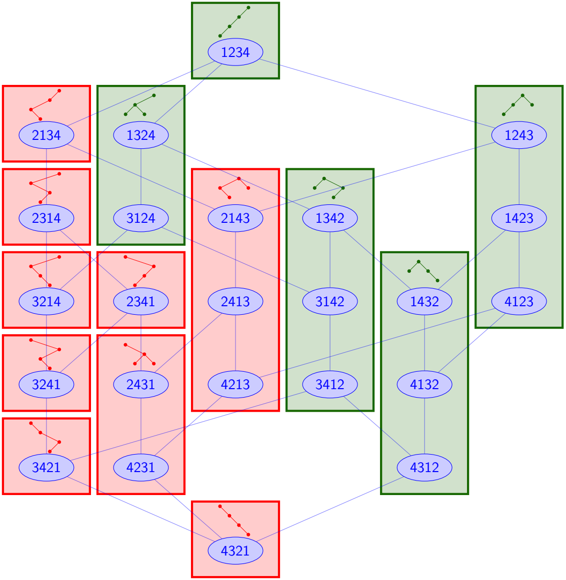
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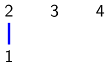
$F_{\leq}(T')$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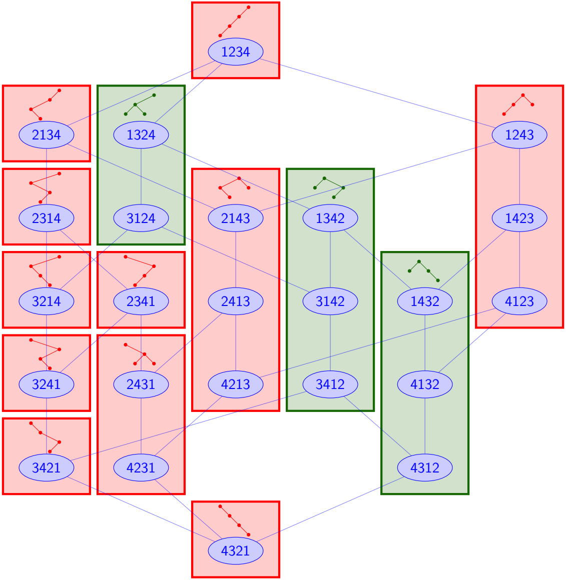
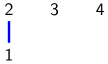


$F_{\geq}(T)$

$F_{\leq}(T')$



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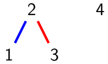

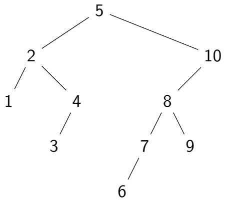
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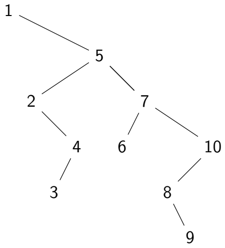
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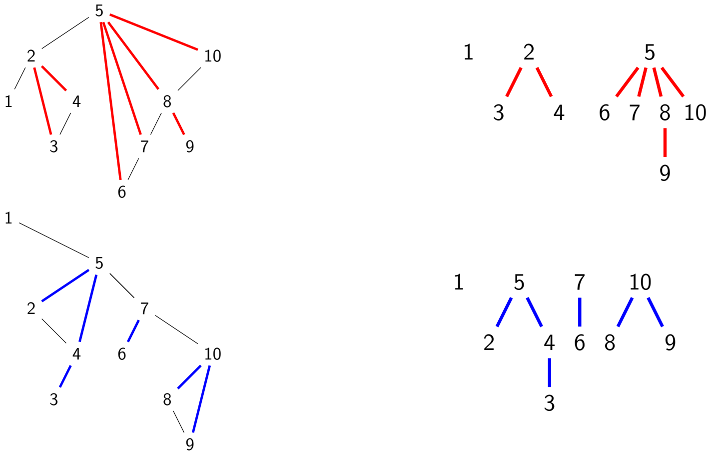

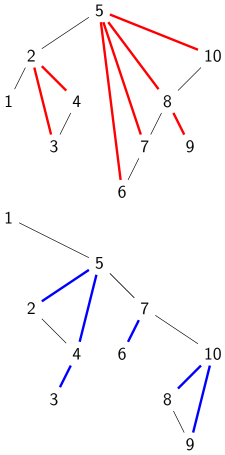
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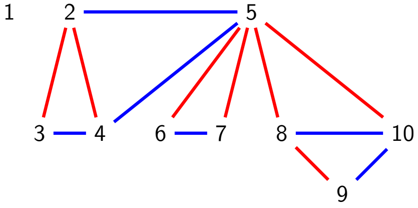
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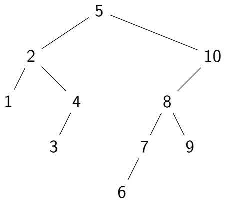
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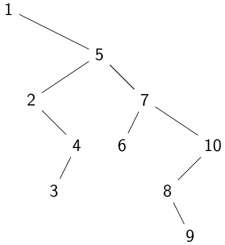
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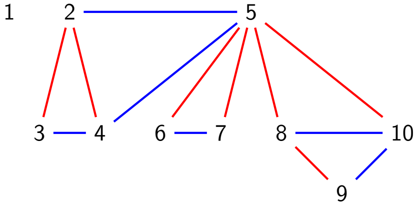
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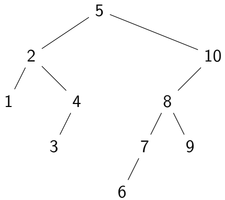

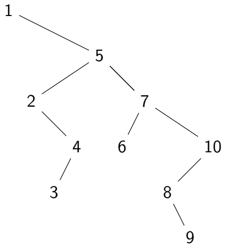

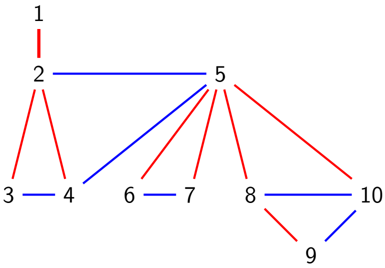

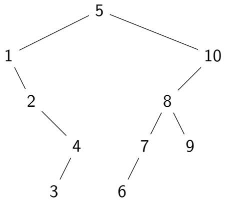
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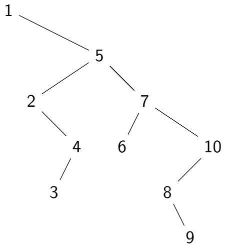
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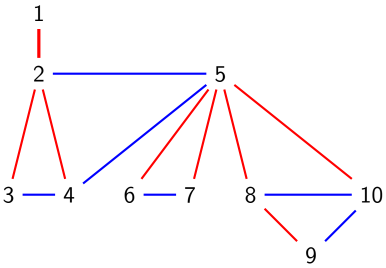
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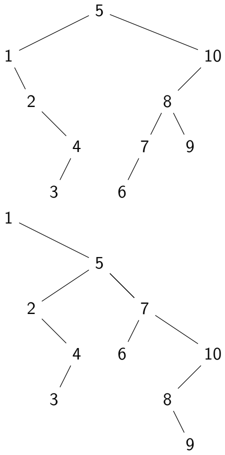
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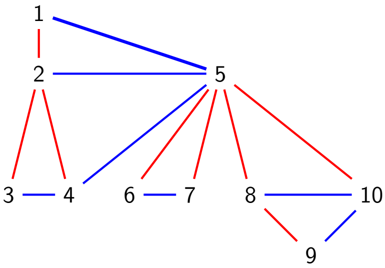
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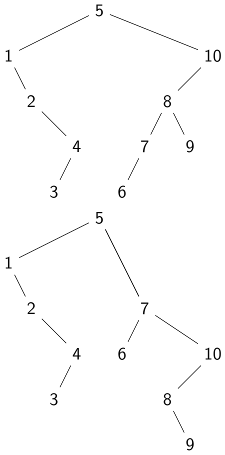
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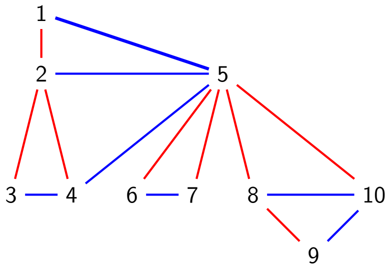
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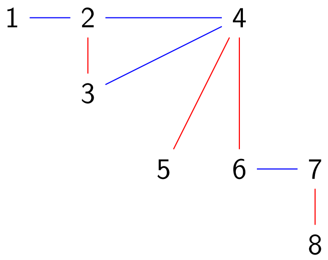
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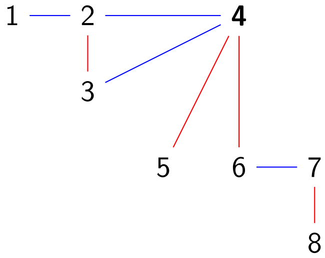
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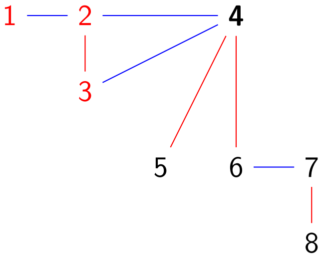
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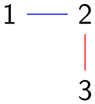
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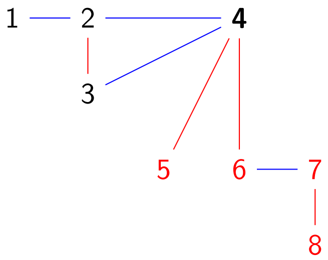
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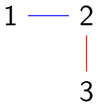
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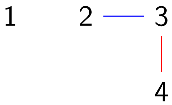
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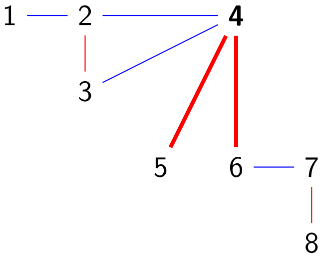

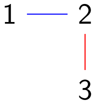

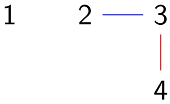

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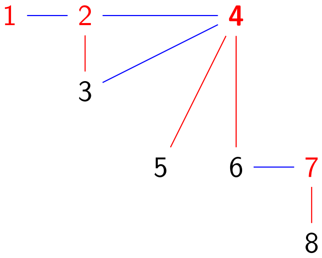
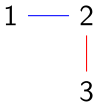
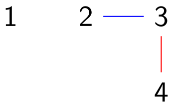


x^4





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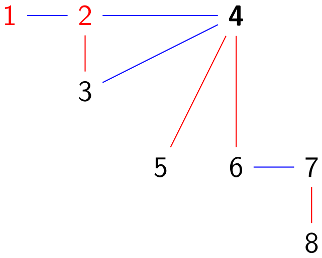
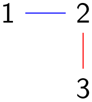
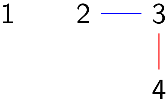


$$x^4 = x^2 . x . \frac{x^3}{x^2}$$



x^2



x^3

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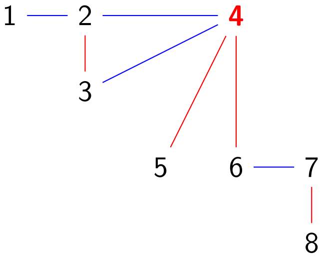
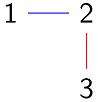
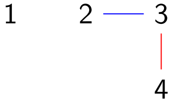


$$x^4 = x^2 . {\color{red}x} . \frac{x^3}{x^2}$$



x^2



x^3

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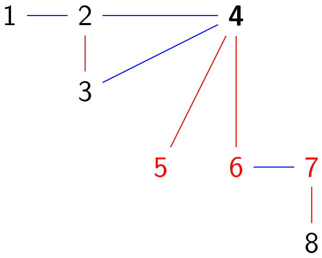
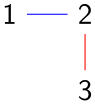
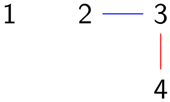




x^2



x^3

$$x^4 = x^2.x.\frac{x^3}{x^2}$$

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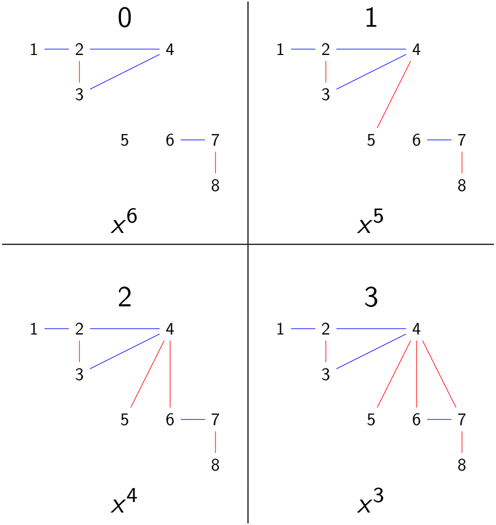


$$= x^3.x.(1 + x + x^2 + x^3)$$

Theorem (Chapoton)

La série génératrice des intervalles de Tamari vérifie

$$\Phi(x,y) = \mathrm{B}(\Phi,\Phi) + 1$$

où

$$\mathrm{B}(f,g) = xyf(x,y)\frac{xg(x,y) - g(1,y)}{x-1}$$

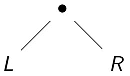
Polynômes de Tamari

On définit récursivement $\mathcal{B}_T$ par

$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

avec $T =$



Theorem (Châtel, P.)

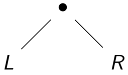
$\mathcal{B}_T$ compte le nombre d'arbres inférieurs ou égaux à T dans le treillis de Tamari en fonction du nombre de nœuds sur leur branche gauche.

Polynômes de Tamari

On définit récursivement $\mathcal{B}_T$ par

$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

avec T = 

Theorem (Châtel, P.)

$\mathcal{B}_T$ compte le nombre d'arbres inférieurs ou égaux à T dans le treillis de Tamari en fonction du nombre de nœuds sur leur branche gauche.

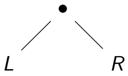
Polynômes de Tamari

On définit récursivement $\mathcal{B}_T$ par

$$\mathcal{B}_\emptyset := 1$$

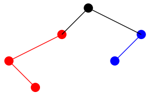
$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

avec $T =$



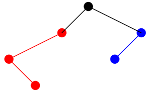
Theorem (Châtel, P.)

$\mathcal{B}_T$ compte le nombre d'arbres inférieurs ou égaux à T dans le treillis de Tamari en fonction du nombre de nœuds sur leur branche gauche.



$$\mathcal{B}_\emptyset := 1$$

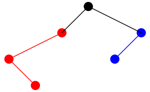
$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$



$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

$$\mathcal{B}_L(x) = x^3 + x^2$$

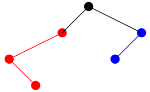


$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

$$\mathcal{B}_L(x) = x^3 + x^2$$

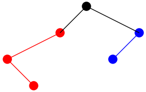
$$\mathcal{B}_R(x) = x^2$$



$$\mathcal{B}_\emptyset := 1$$

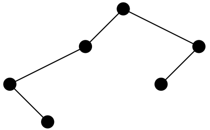
$$\mathcal{B}_T(x) := x(x^3 + x^2)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$

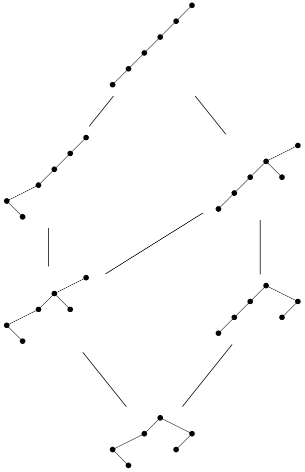
$$\mathcal{B}_R(x) = x^2$$



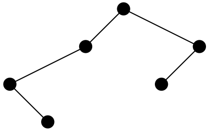
$$\mathcal{B}_{\emptyset} := 1$$

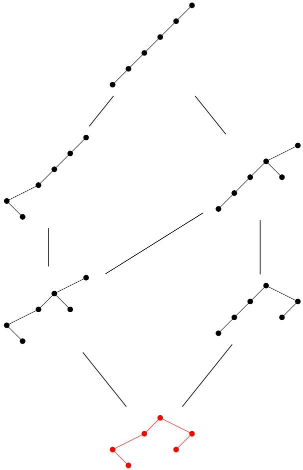
$$\mathcal{B}_T(x) := x(x^3 + x^2)(1 + x + x^2)$$



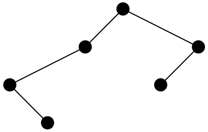


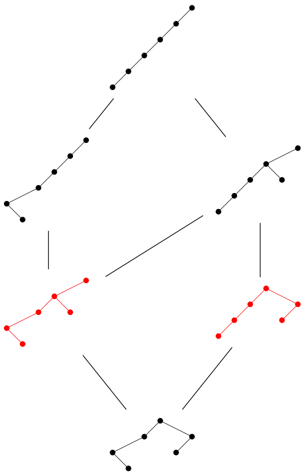
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



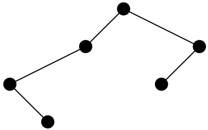


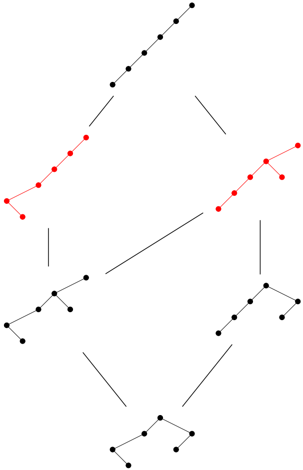
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



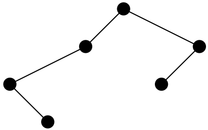


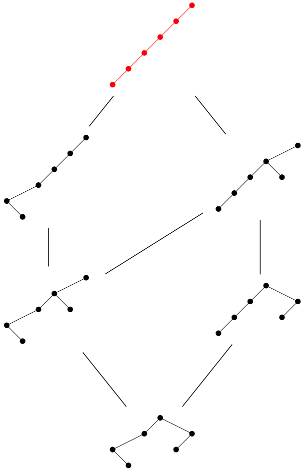
$$\mathcal{B}_T(x) = x^3 + {\color{red}2x^4} + 2x^5 + x^6$$



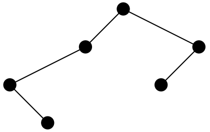


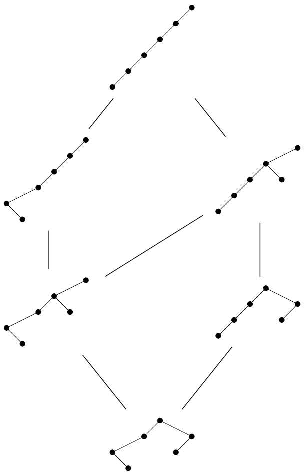
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$





$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$





$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$
$$\mathcal{B}_T(1) = 6$$

Quelques autres résultats et pistes...

- Lien avec les flots sur les arbres.
- Bijection avec les triangulations.
- Nouvelle involution sur les intervalles.
- Généralisation à m-Tamari.
- Mieux comprendre le treillis de Tamari et ses liens multiples avec d'autres théories ?