

# Des relations binaires au treillis de Tamari

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Université Paris-Sud

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## Le tri à bulle sur les permutations


251436

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251436  
215436

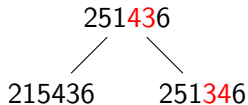


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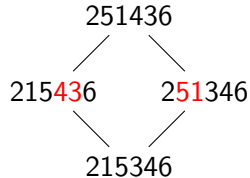
251436  
215436



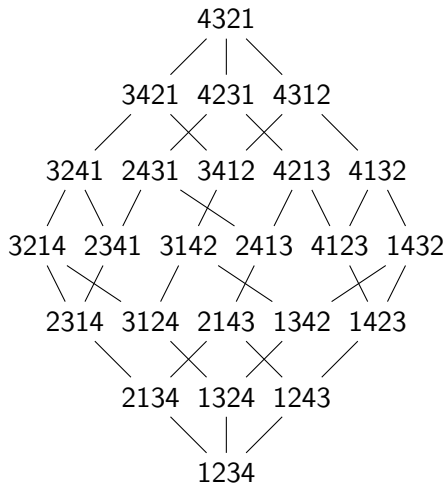
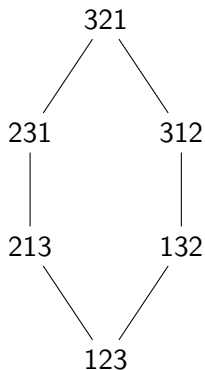
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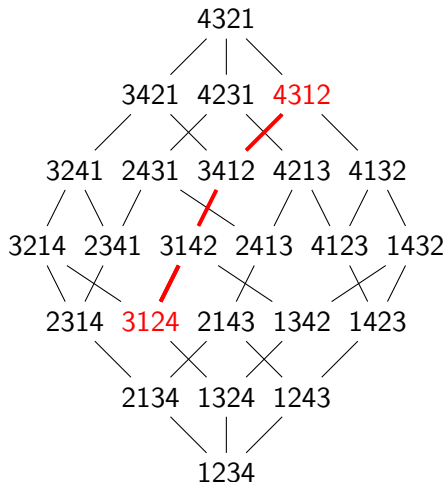
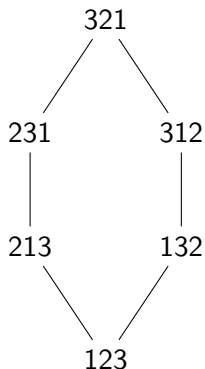


# Ordre faible

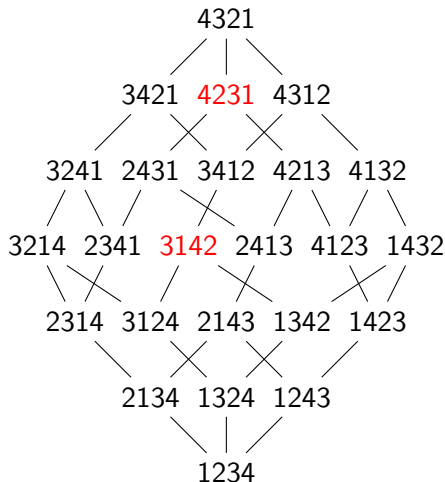
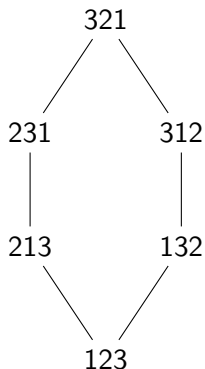




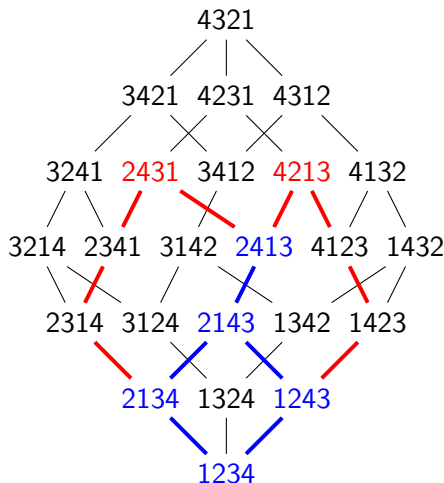
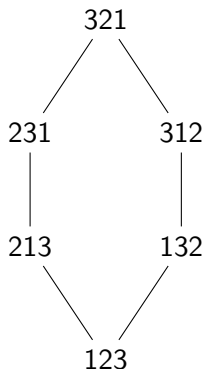
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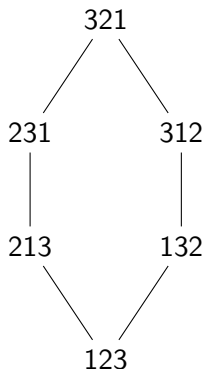


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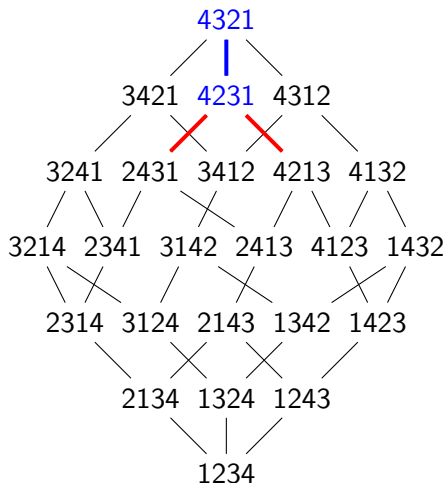


$$2413 \wedge 4213 = 2413$$

# Ordre faible



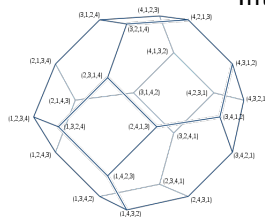
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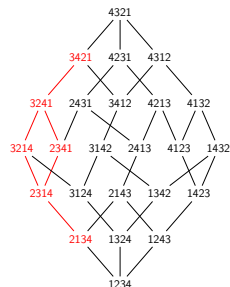
$$2413 \vee 4213 = 4231$$

## Triple interprétation

Géométrie



## Interprétation combinatoire



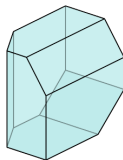
Algèbre

$$F_{21} \cdot F_{12} = F_{21 \sqcup 12}$$

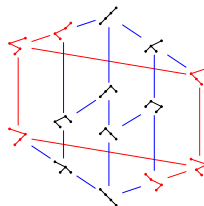
$$= F_{2134} + F_{2314} + F_{2341} + F_{3214} + F_{3241} + F_{3421}$$

## Triple interprétation

Géométrie



Interprétation combinatoire



Algèbre

$$P \cdot P = P + P + P + P + P + P$$

## Graphe d'une permutation

4312

## Graphe d'une permutation

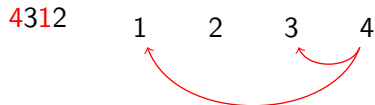
4312



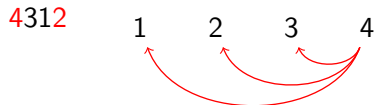
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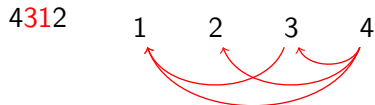
## Graphe d'une permutation



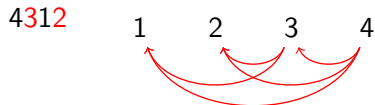
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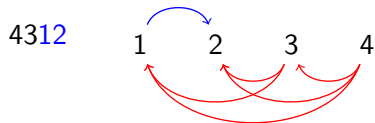
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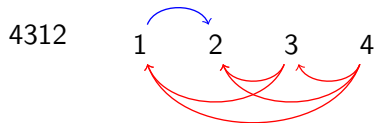
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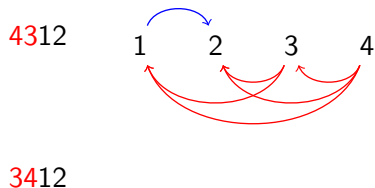
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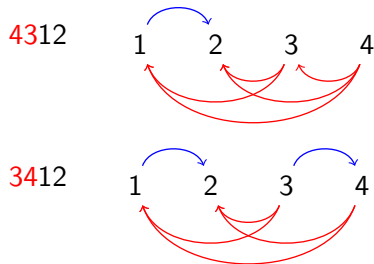


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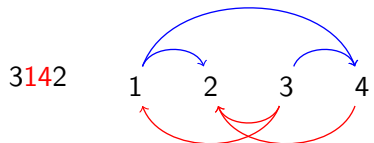
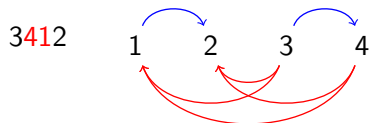
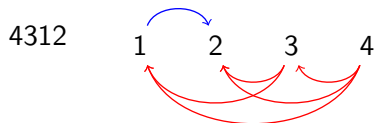




## Graphe d'une permutation



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## Relations binaires sur les entiers

Soit  $R$  une relation de taille  $n$ .

1      2      ...       $i$       ...       $j$       ...       $k$       ...       $n$

## Relations binaires sur les entiers

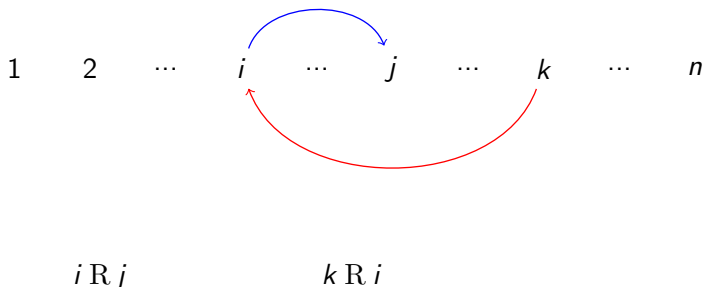
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$$i R j$$

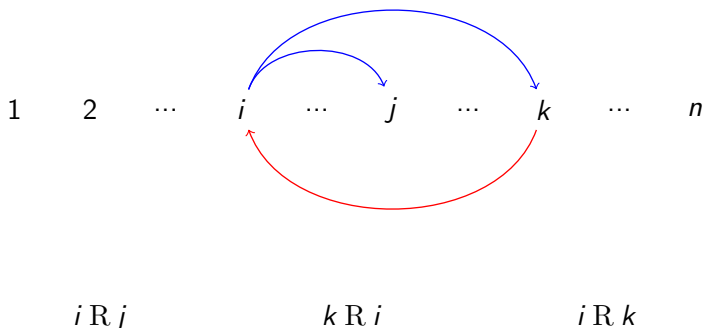
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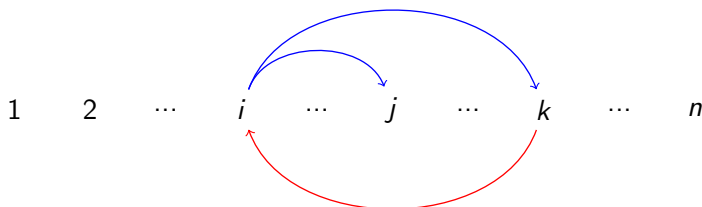
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$$i R j$$

$$k R i$$

$$i R k$$

En taille  $n$ ,  $2^{n(n-1)}$  relations binaires possibles.

## Ordre partiel sur les relations

Soit  $R$  une relation binaire

$$R^{\text{Inc}} = \{i R j, i < j\}$$

$$R^{\text{Dec}} = \{j R i, i < j\}$$



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Soit  $R$  une relation binaire

$$R^{\text{Inc}} = \{i R j, i < j\}$$

$$R^{\text{Dec}} = \{j R i, i < j\}$$

Soient  $R$  et  $S$  deux relations binaires,

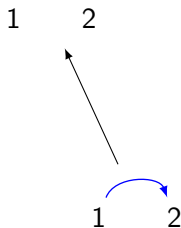
$$R \preceq S \Leftrightarrow R^{\text{Inc}} \supseteq S^{\text{Inc}} \text{ et } R^{\text{Dec}} \subseteq S^{\text{Dec}}$$

## Relations de taille 2

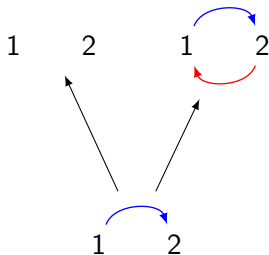
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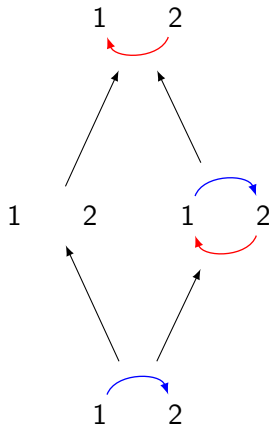
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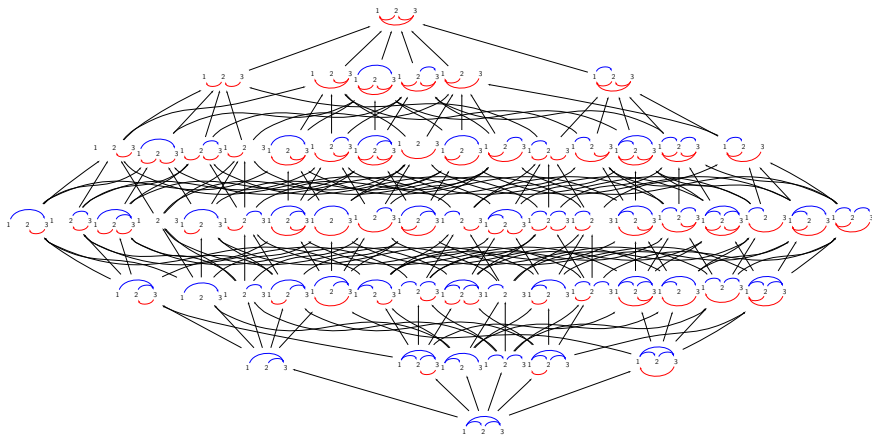


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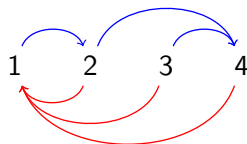
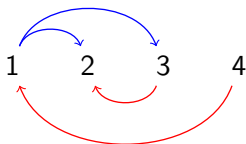


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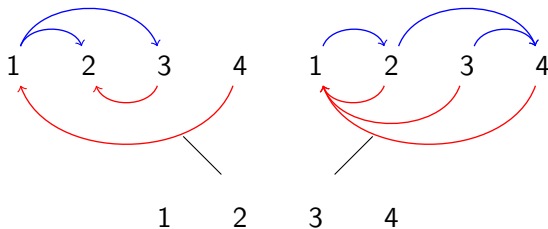


# Inf et Sup

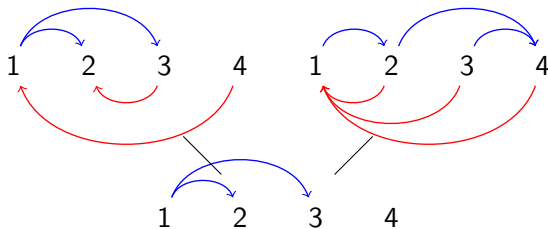




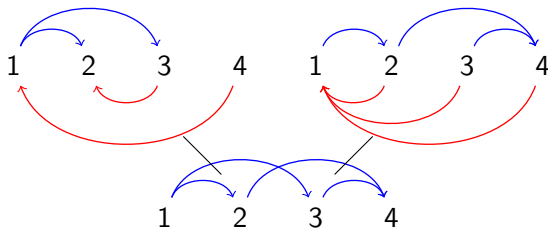
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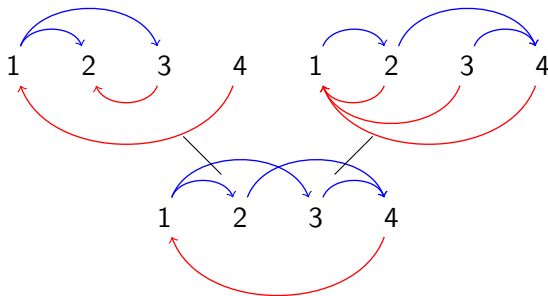
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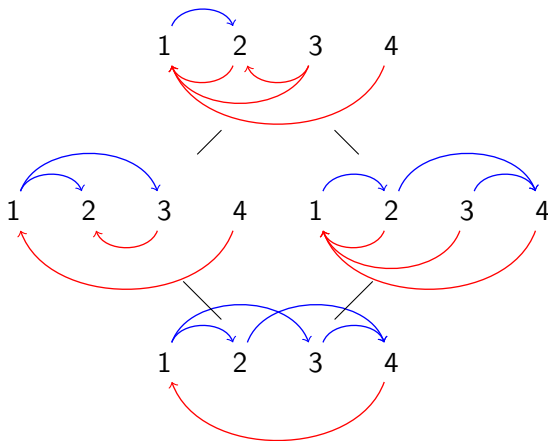
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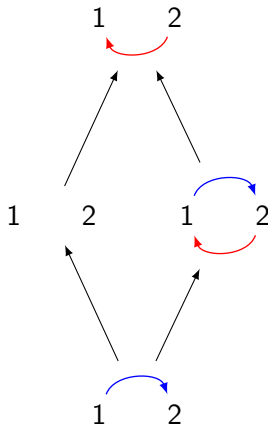


On veut conserver les relations

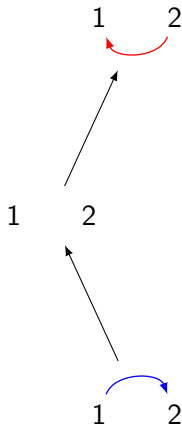
- ▶ antisymétriques
- ▶ transitives

(posets)

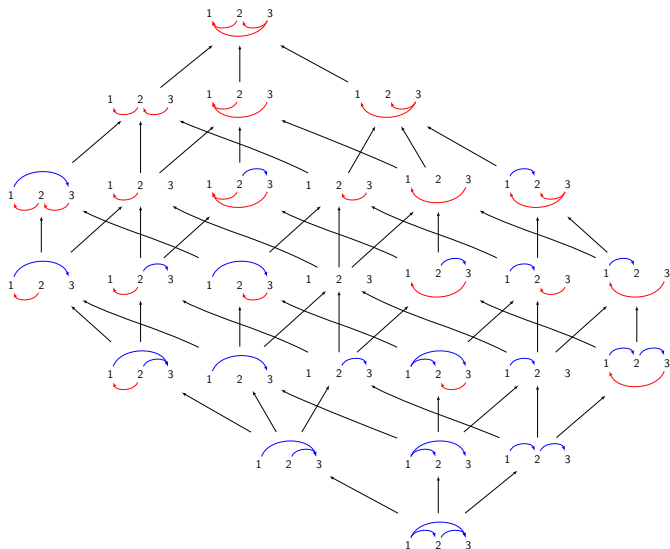
## Antisymétrie



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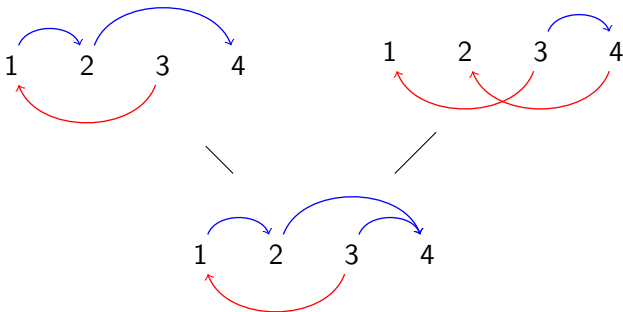






## Sous treillis ?

Si  $R$  et  $S$  sont antisymétriques, est-ce que  $R \wedge S$  l'est aussi ?



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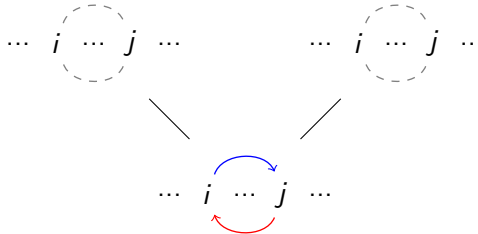
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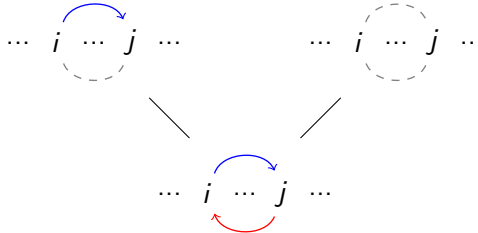
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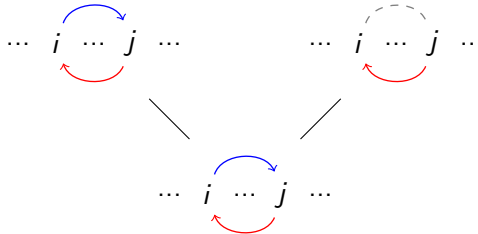
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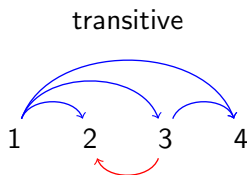
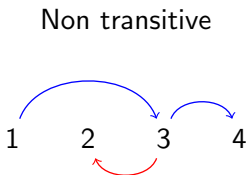


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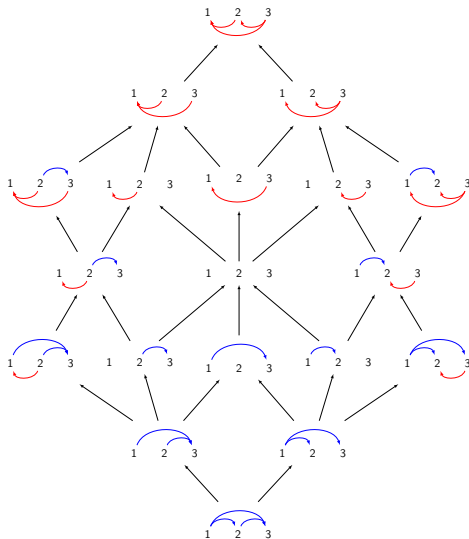
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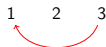
## Transitivité



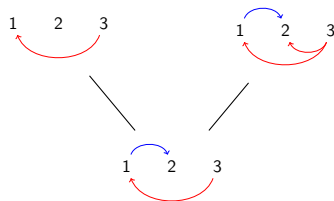
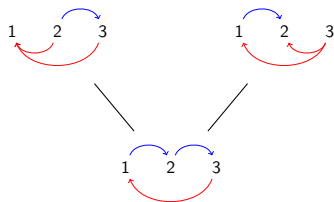




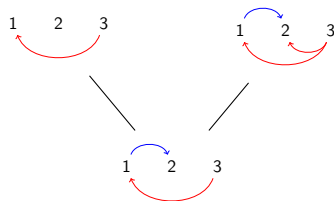
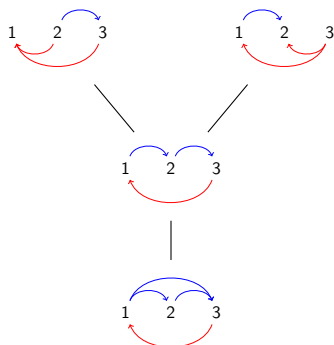
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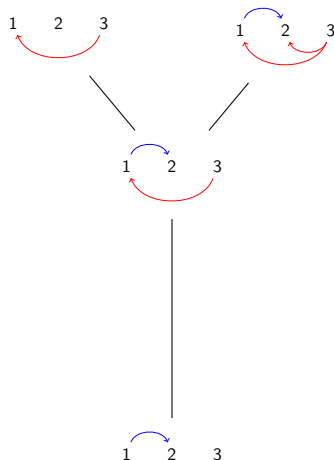
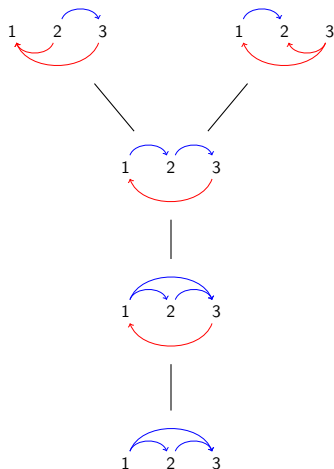
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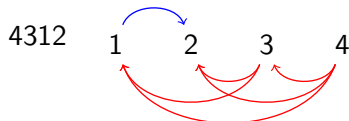
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## Retour aux permutations

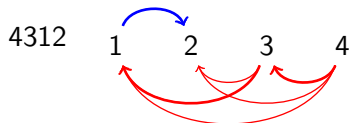


On a  $i R j$  ssi le nombre  $i$  est placé avant le nombre  $j$  dans la permutation.

La relation est donc

- ▶ antisymétrique
- ▶ transitive
- ▶ **et totale**

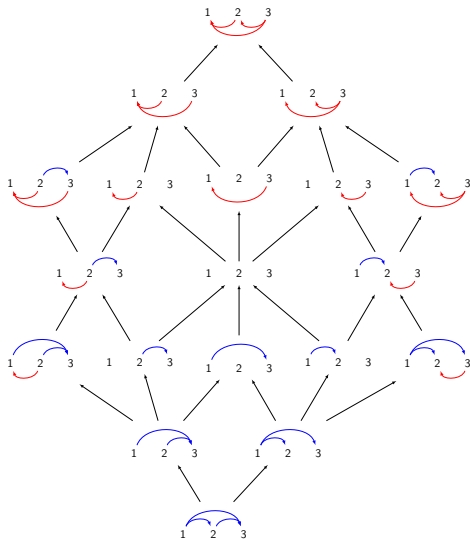
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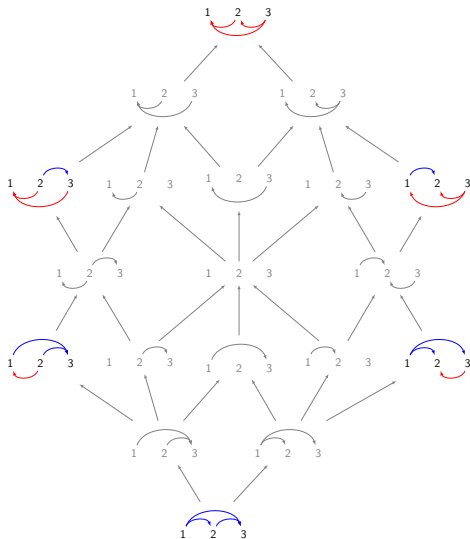
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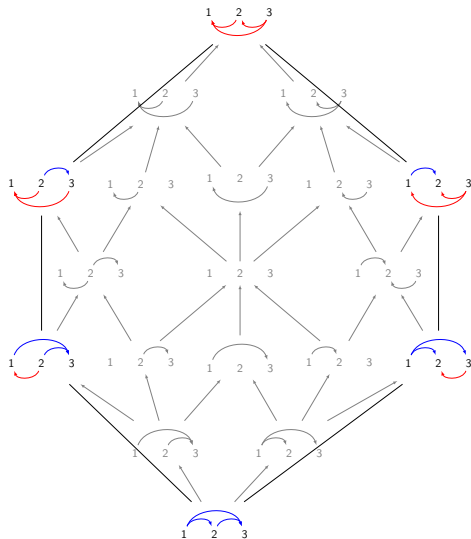
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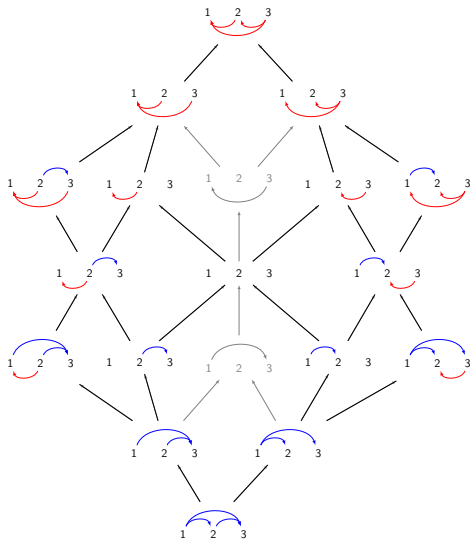
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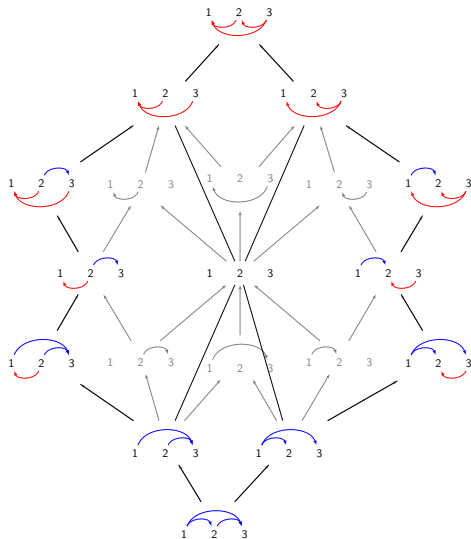


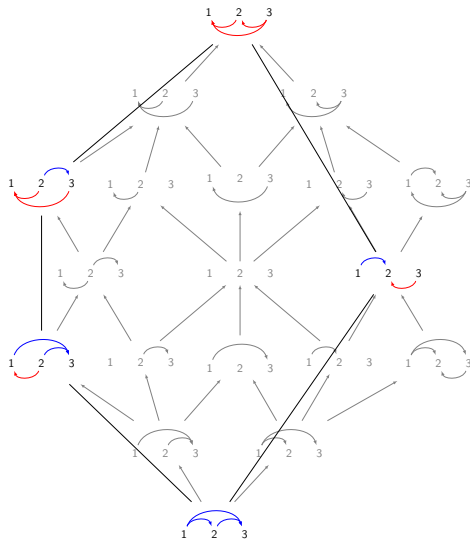


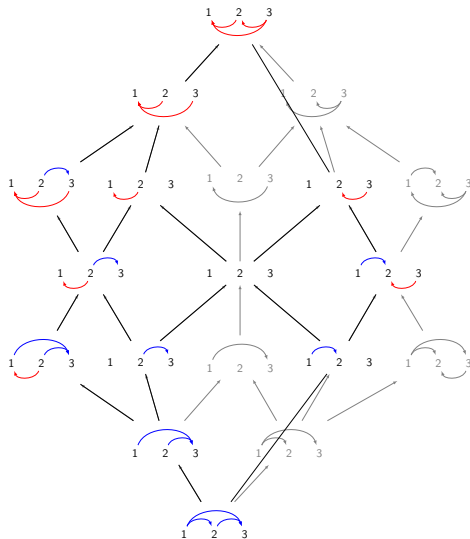


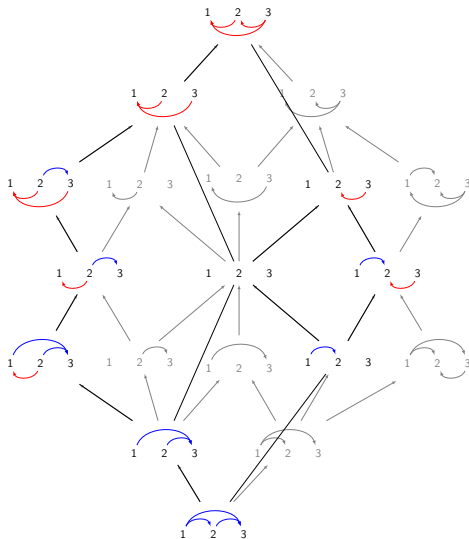


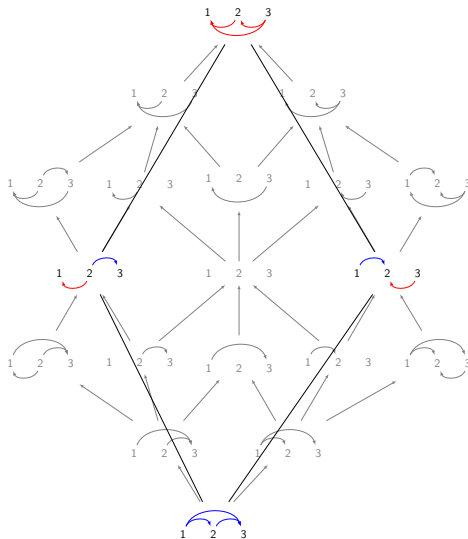




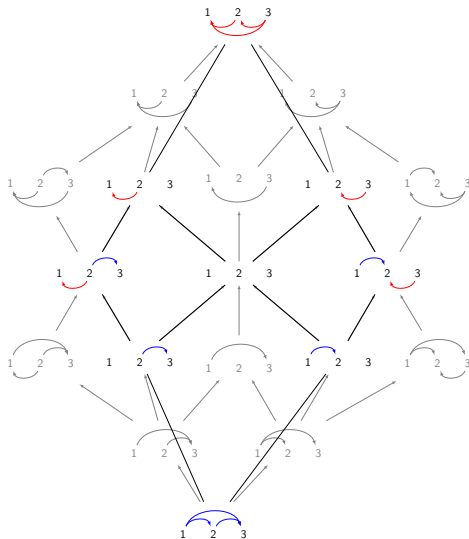












## Travail en cours

- ▶ Prouver que tous ces sous-ordres sont des treillis.
- ▶ Le cas échéant, prouver que ce sont des sous-treillis.

## Et après...

- ▶ Étudier les algèbres associées.